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MINIMUM TIME TURNS CONSTRAINED
TO THE VERTICAL PLANE

THESIS

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MINIMUM TIME TURNS CONSTRAINED

TO THE VERTICAL PLANE

THESIS

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LIST OF SYMBOLS

A - unknown parameter vector

C_D - drag coefficient

C_{D_o} - parasite drag coefficient

C_L - lift coefficient

C_{L_α} - lift curve slope

D - drag

F - augmented performance index

G - performance index (t_f)

g - gravitational acceleration

H - Variational Hamiltonian

h - altitude

K - induced drag parameter

L - lift

M - final condition vector

N - side force

S - wing area

T - thrust

t - time

U - control vector

V - velocity

W - weight

X - state vector

x - distance (x-direction)

y - distance (y-direction)

α - angle of attack

ε - thrust angle of attack

ξ - thrust side slip angle

Θ - flight path angle

λ - Lagrange multiplier vector (differential equations)

ν - Lagrange multiplier vector (final conditions)

π - thrust control variable

ρ - density

σ - density ratio

τ - non-dimensional time

ϕ - bank angle

ψ - heading angle

$\dot{a}$ - da/dt (a arbitrary)

a^T - a transposed (a arbitrary)

$(\ )_i$ - initial value

$(\ )_f$ - final value

$(\ )_c$ - corner value

$(\ )_A$ - $(\ )/A$ (A arbitrary)

$(\ )_o$ - ( ) at sea level

Abstract

The objective of this study is to find the throttle controls and trajectories which yield the minimum turning times for a high performance aircraft with thrust reversal capability. The aircraft remains in the vertical plane allowing only pull-up and split-s maneuvers. A second-order parameter optimization method coupled with the suboptimal control approach is used to solve the minimum time-to-turn problem.

The results of the study found that trajectories restricted to the vertical plane gave different results, and in at least one case better results, than those not so constrained. The results also indicate that depending on the maneuver performed, thrust reversal is beneficial in reducing the minimum time-to-turn regardless of whether the aircraft's initial velocity is above or below corner speed. Finally, the results demonstrate that thrust reversal can be utilized for minimum time turns with resulting increases in specific energy.

MINIMUM TIME TURNS CONSTRAINED

TO THE VERTICAL PLANE

I. Introduction

Background

In recent years, great interest has been taken in the minimum time-to-turn capability of high performance fighter type aircraft. This interest has been stimulated by the air-to-air combat requirements demanded of these aircraft. Such requirements include quicker and tighter turns, commanded nose pointing capabilities, and increased responsiveness to the requirements of expanded weapon system envelopes. Recent technological developments have contributed substantially to the efforts of improving the time required for aircraft turns. These developments include thrust to weight ratios greater than one, inflight thrust reversal capabilities, increased angle of attack capabilities, and greater structural load factors. With thrust to weight (T/W) ratios greater than one, aircraft are now capable of accelerating in pull-up maneuvers. Coupled with inflight thrust reversers, higher T/W ratios contribute to turning performance because they allow the aircraft to maintain corner velocity even with changes in altitude. Lastly, both the improved angle of attack capabilities and load factor enhancement aid in reducing the time to turn by providing increased lift throughout the maneuver.

Over the years, numerous studies have addressed the minimum time-to-turn problem. Each of these studies has had its own unique problem solving approach resulting in some interesting conclusions. In the most recent past, Humphreys, Hennig, Bolding, and Helgeson in Ref (1), Johnson in Ref (2), and Peterson in Ref (3) obtained results for minimum time turns for aircraft with and without inflight thrust reversal capabilities. In each of the studies, the aircraft trajectories were unconstrained. The unconstrained aircraft equations of motion are not integrateable for solutions confined to the vertical plane because the flight path angle causes a singularity as it passes through 90 degrees. Also, when the trajectories do not lie in the vertical plane, the lift vector is not utilized to its fullest potential. For these reasons, minimum time turns constrained to the vertical plane are studied in the belief that the planar constrained trajectory will yield the minimum turning time due to the planar alignment of the lift, thrust, and gravity vectors.

Statement of the Problem

The problem is to find the optimal throttle controls and the resulting trajectories for a high performance fighter aircraft performing minimum time turns constrained to the vertical plane. The vertical plane restriction limits the aircraft to only the pull-up and the split-s maneuvers. The maneuvers will begin at a given set of initial conditions and will be completed when a given set of end conditions have been met. Completion of the turning maneuver in minimum time is of paramount importance. Throughout the maneuver the aircraft will be subjected to typical limitations including: 1) aircraft maximum angle of attack, 2) aircraft structural limitations, and 3) minimum and maximum throttle limitations. For each of these trajectories the throttle controls will be varied, while all other aircraft characteristics remain the same.

For this study, the identical aircraft as used in Ref (2) is chosen to perform the prescribed maneuvers. In order to closely compare with the results of the Johnson study, the same optimization technique (the suboptimal control approach to the optimal control problem) will be employed.

Assumptions

In order to properly model the aircraft equations of motion, the aerodynamic and thrust forces, and the atmosphere, certain simplifying assumptions must be made. The aircraft maneuvers over a flat earth with a constant gravitational acceleration. The aircraft consumes negligible fuel during the performance of the maneuvers, the thrust angle of attack equals the aircraft angle of attack. These assumptions are typical when describing the motion of a point mass aircraft. The lift coefficient is a linear function of angle of attack up to the maximum angle of attack, and the drag polar is a parabolic function. Due to the small change in the density ratio as a result of changes in altitude during turns, the maximum thrust available throughout the turn is constant. The aircraft performs in the Standard Atmosphere as defined by NASA in Ref (4). Finally, it is assumed that for purposes of modeling the aircraft maneuvers, the aircraft thrust control can change instantaneously. This assumption is considered valid since this type of technology is in the developmental stages.

The following sections will discuss the formulation of the constrained minimum time-to-turn problem. Section II defines the minimum time-to-turn problem in three dimensions and the particular case of a turn in the vertical plane. Section III details the simplifying characteristics of the suboptimal control approach to the optimal control problem. Section IV outlines the application of the suboptimal control approach to the minimum time-to-turn problem. Finally, Sections V and VI will discuss the results of the study and conclusions, respectively.

II. Problem Definition

This section defines the minimum time-to-turn problem. It examines the turning maneuvers and the equations of motion which generate those maneuvers. The aircraft characteristics, control variable constraints, and atmosphere model complete the problem definition. The following paragraphs discuss each of these items in depth.

The Maneuver

The aircraft turns are constrained to the vertical plane. This limits the aircraft to two basic maneuvers, the pull-up and the split-s. The split-s maneuver is accomplished by rolling the aircraft to the inverted position and performing an inverted pull-up. Each maneuver has two initial velocities; one case above corner velocity, and the second below the corner velocity. The four maneuvers will be referred to throughout the remainder of the thesis as the split-s(low velocity) case, the split-s (high velocity) case, the pull-up (low velocity) case, and the pull-up (high velocity) case. Each of the maneuvers begins with the aircraft flying straight and level ($\Theta_i=0$). The maneuver is complete when the aircraft once again has reached straight and level flight ($\Theta_f=0$). The initial and final maneuver conditions appear in Table 1. Also appearing in Table 1 are two data sets chosen from the Johnson study for data comparison purposes. Because all of the cases begin at an altitude of 13990 ft, the gravitational acceleration at that altitude, $g=32.131$ ft/sec^2, Ref (4), will be used throughout the maneuvers.

Maneuver	V_i (ft/sec)	h_i (ft)	θ_i	θ_f
Split-s(LV)	621	13990	0	0
Split-s(HV)	903	13990	0	0
Pull-up(LV)	621	13990	0	0
Case 4	621	13990	0	0
Pull-up(HV)	903	13990	0	0
Case 5	903	13990	0	0

Table 1. Initial and Final Conditions

Aircraft Equations of Motion

The equations of motion for a point mass aircraft in flight over a flat earth as derived in Ref(5:48-49) are:

$$\dot{X} = V\cos\Theta\cos\Psi \tag{1}$$

$$\dot{Y} = V\cos\Theta\sin\Psi \tag{2}$$

$$\dot{h} = V\sin\Theta \tag{3}$$

$$\dot{V} = g\left(\frac{T}{W}\cos\epsilon\cos\xi - \frac{D}{W} - \sin\Theta\right) \tag{4}$$

$$\dot{\Theta} = \frac{Qg\sin\phi}{WV} + \frac{gL\cos\phi}{WV} - \frac{g\cos\Theta}{V} - \frac{Tg}{WV}\left(\sin\phi\cos\epsilon\sin\xi - \cos\phi\sin\epsilon\right) \tag{5}$$

$$\dot{\psi} = \frac{g\sin\phi}{V\cos\Theta}\left(\frac{T}{W}\sin\epsilon + \frac{L}{W}\right) \tag{6}$$

Assuming that the maneuver performed is a coordinated turn allows Q=0. Additional assumptions are that the thrust angle of attack is equal to the aircraft angle of attack, and that the angle of attack is small. These assumptions imply that $\sin\xi=0$, $\cos\xi=1$, $\varepsilon=\alpha$, $\sin\alpha=\alpha$, and $\cos\alpha=1$. The equations of motion then become:

$$\dot{x} = V\cos\Theta\cos\Psi \tag{7}$$

$$\dot{y} = V\cos\Theta\sin\Psi \tag{8}$$

$$\dot{h} = V\sin\Theta \tag{9}$$

$$\dot{V} = g\left(\frac{T}{W} - \frac{D}{W} - \sin\Theta\right) \tag{10}$$

$$\dot{\Theta} = \frac{g}{V}\left[\left(\frac{T}{W}\alpha + \frac{L}{W}\right)\cos\phi - \cos\Theta\right] \tag{11}$$

$$\dot{\Psi} = \frac{g\sin\phi}{V\cos\Theta}\left(\frac{T}{W}\alpha + \frac{L}{W}\right) \tag{12}$$

The flight of a point mass aircraft is now constrained to the vertical plane to allow only the pull-up and the split-s maneuvers. This constraint leads to further simplification of the equations of motion. For flight in the vertical plane, there is no motion in the x-y plane indicating that y=constant and $\dot{y}=0$. Similarly, the aircraft is not permitted longitudinal maneuvers out of the x-h plane which forces $\phi=0$, $\Psi=0$, $\sin\phi=0$, $\sin\Psi=0$, $\cos\phi=1$, and $\cos\Psi=1$. The equations of motion are now reduced to the desired form for flight in the vertical plane:

$$\dot{x} = V\cos\Theta \tag{13}$$

$$\dot{h} = V\sin\Theta \tag{14}$$

$$\dot{V} = g\left(\frac{T}{W} - \frac{D}{W} - \sin\Theta\right) \tag{15}$$

$$\dot{\Theta} = \frac{g}{V}\left(\frac{T}{W}\alpha + \frac{L}{W} - \cos\Theta\right) \tag{16}$$

The split-s maneuver is generated by placing a negative angle of attack (implying ϕ=180 degrees) into these equations. These equations model the motion of the aircraft with respect to an earth fixed coordinate frame. The state variables have been reduced to x, h, V, and Θ. The control variables for the maneuver are now limited to α and T.

Aircraft Characteristics

The aircraft chosen to perform the selected maneuvers is identical to that used in previous work, Ref (2), for the purpose of data comparison. The characteristics of the aircraft which require development are the lift-to-weight, drag-to-weight, and the thrust-to-weight ratios. Lift and drag are the two aerodynamic forces in the equations of motion. These forces can be expressed in terms of atmospheric and aircraft parameters as:

$$L = \frac{\rho_0 \sigma V^2 S C_L}{2} \tag{17}$$

$$D = \frac{\rho_0 \sigma V^2 S C_D}{2} \tag{18}$$

The lift and drag coefficients are:

$$C_L = C_{L_\alpha}\alpha \tag{19}$$

$$C_D = C_{D_0} + KC_L^2 \tag{20}$$

In the equations of motion, the lift and drag forces are expressed as non-dimensional lift-to-weight and drag-to-weight ratios. When equations (19) and (20) are incorporated into equations (17) and (18) these ratios become:

$$\frac{L}{W} = \frac{\rho_0 \sigma V^2 S}{2W} C_{L_\alpha}\alpha \tag{21}$$

$$\frac{D}{W} = \frac{\rho_0 \sigma V^2 S}{2W}\left(C_{D_0} + KC_L^2\right) \tag{22}$$

The aircraft parameters can be substituted into these equations in order to obtain numerical values. These aircraft parameters are:

C_{L_α}=5.0	S=237 sq ft
C_{D_0} =.02	α_{max} = .2 radians
K =.05	$(T/W)_{max}$ = 1.5
W =12,150 lb	$(L/W)_{max}$ = 7.22

The thrust-to-weight ratio (T/W) is also required for use in the equations of motion. It is represented as a function of the maximum thrust available (T_{max}) and a thrust control variable, (π).

The thrust-to-weight ratio becomes:

$$\frac{T}{W} = \frac{T_{MAX}}{W} \pi = \left(\frac{T}{W}\right)_{MAX} \pi \tag{23}$$

where π has limits of -.6 and 1.0.

<u>Aircraft/Control Variable Constraints</u>

The aircraft is subject to three basic physical constraints during its vertical plane maneuvers. These constraints include a maximum angle of attack, a maximum structural load factor, and minimum and maximum thrust limitations. These constraints are:

$$\alpha \leq \alpha_{MAX} \tag{24}$$

$$\frac{L}{W} \leq \left(\frac{L}{W}\right)_{MAX} \tag{25}$$

$$T_{MIN} \leq T \leq T_{MAX} \tag{26}$$

The thrust control constraint, equation (26), can also be expressed in terms of the thrust control variable:

$$\pi_{MIN} \leq \pi \leq \pi_{MAX} \tag{27}$$

<u>The Atmosphere</u>

The atmosphere model assumed in the solution of the problem has density ratio, σ, expressed as:

$$\sigma = \frac{\rho}{\rho_0} = \left[1 - \left(\frac{N-1}{N}\right)\frac{g_0}{RT_0} h\right]^{\frac{1}{N-1}} \tag{28}$$

where

$$\rho_o = .002378 \text{ SLUGS/FT}^3$$

$$g_o = 32.174 \text{ FT/SEC}^2$$

$$T_o = 518.688 \text{ °R}$$

$$N = 1.235$$

$$R = 1715 \text{ FT}^2\text{/SEC}^2\text{-°R} \; .$$

III. The Suboptimal Control Problem

(Ref (6))

The minimum time-to-turn problem is an optimal control problem. The optimal control approach is an inherently difficult and complex solution method. Because the optimal control problem can be hard to solve, ways of approximating the optimal solution exist. The suboptimal control problem is a distinct simplification to the optimal control problem. The approximation of the control variables reduces the problem to one of parameter optimization. The formulation of the suboptimal control problem is developed in the following paragraphs.

Suboptimal Control Problem Formulation

The suboptimal control formulation requires that the control variables be approximated by a mathematical function which is competely characterized by a finite number of parameters. This approximation method assumes that some knowledge of the type of control required is available. The parameters describing this mathematical function will be expressed as the vector B. The control variables are then expressed as

$$U = U(t, B) \tag{29}$$

where U can be any functional form. If A denotes the vector of unknown parameters,

$$A = [t_f, B]^T \tag{30}$$

then the integration of the differential equations (equations of motion), (13) thru (16), subject to the prescribed initial, final, and control variable constraints leadsto the functional relation,

$$X_f = X_f(A) \quad . \tag{31}$$

Because the performance index and final condition constraints depend only on t_f and X_f, and hence A, the optimal control problem has been reduced to one of parameter optimization. The problem now consists of merely finding the unknown parameter vector A which minimizes the performance index,

$$G = G(A) = t_f \tag{32}$$

subject to the final condition constraints

$$M(A) = 0 \quad . \tag{33}$$

Solution of the Suboptimal Control Problem

If the augmented performance index, F, is defined as

$$F(A, \nu) = G(A) + \nu^T M(A) \tag{34}$$

then the only constraints which the vectors A and ν must satisfy are the result of requiring the first variation to be zero. These conditions are:

$$F_A^T(A,\nu) = 0 \tag{35}$$

$$M(A) = 0 \tag{36}$$

Hull and Edgeman, Ref (6), formulate a second-order parameter optimization technique specifically for application to the sub-optimal control problem. The technique uses second derivative information in order to determine how to change the parameter vector A and the Lagrange multiplier ν such that the first variation conditions are driven to zero. These changes to A and ν are:

$$\delta\nu = \left(M_A F_{AA}^{-1} M_A^T\right)^{-1}\left(-P M_A F_{AA}^{-1} F_A^T + QM\right) \tag{37}$$

$$\delta A = -F_{AA}^{-1}\left(P F_A^T + M_A^T \delta\nu\right) \tag{38}$$

where P and Q are scaling factors which are used to control the optimization process. The iterative algorithm for incrementing A and ν is easily programmed as follows:

1. guess A and ν
2. integrate the differential equations and determine X_f
3. compute M, M_A, M_{AA}, F_A, and F_{AA}
4. compute δA and $\delta\nu$ with selected values of P and Q
5. check convergence criteria
6. set $A=A+\delta A$, $\nu=\nu+\delta\nu$ and go to 2

The procedure is greatly simplified over the optimal control problem but it does require some skill in initially choosing ν, P, and Q. Hull and Edgeman, Ref (6), were able to calculate the initial value of ν from a gradient approach

$$\nu = \left(M_A M_A^T\right)^{-1}\left[\left(Q/P\right)M - M_A G_A^T\right]. \quad (39)$$

However, the procedure for selecting P and Q is not as obvious. The constants, P and Q, are initially introduced in the $\delta\nu$ and the δA conputations. The constants ν and A are introduced to control F_A^T and M, so that F_A^T and M (the first variation of the augmented performance index) both go to zero. Hence, one method of selecting P and Q is to choose them such that the norms of F_A^T and M always decrease. Another approach is to select P and Q such that the norms of $\delta\nu$ and δA fall below some criteria. However, F_A^T must still be decreasing continuously. With either method of P and Q selection, it is required that P and Q equal one in the final stages of convergence.

There are two possible convergence criteria. In one case the norms of F_A^T anf M are each below some specified convergence tolerance. The second potential criteria is similar to the first; however, the norms of δA and $\delta\nu$ are compared to their own specified tolerance.

IV. Application of Suboptimal Control to the Minimum Time to Turn Problem

Adaptation of the suboptimal control algorithm, as outlined in Section III, to the minimum time to turn problem is a relatively straightforward procedure. In order to prepare the problem for solution on the computer, a number of modifications must be completed. First of all, the aircraft equations of motion (differential equations) must be manipulated into a numerically integrateable form. Similarly, the mathematical form of the control variables must be defined, and the control variable constraints must be incorporated into the solution process. The numerical methods utilized for first and second-order derivative information must also be specified. Lastly, the convergence criteria and initialization of the unknown vector A must be established.

Aircraft Equations of Motion

Because the final time of the maneuver is not initially fixed, integration of the differential equations, (13) thru (16), with respect to non-dimensional time simplifies the numerical integration process. Therefore, the equations will be integrated over the interval

$$0 \leq \tau \leq 1 \tag{40}$$

where

$$\tau = \frac{t}{t_f} \tag{41}$$

The transformation of the differential equations into a non-dimensional form is accomplished through the use of the deivative chain rule. This applicationyields

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\frac{1}{t_f} \qquad (42)$$

and finally

$$\frac{dx}{d\tau} = t_f \dot{x} \qquad (43)$$

Therefore, the aircraft equations of motion, (13) thru (16), can be transformed to non-dimensional form simply by multiplying the right side of the equation by the final time, t_f.

The known aircraft and atmospheric parameters can be substituted into equations (21), (22), (23) after they are non-dimensionalized. Then, the constant of gravitational acceleration and equations (21), (22), and (23) are substituted into the aircraft equations of motion, yielding the differential equations as a function of τ:

$$\frac{dx}{d\tau} = t_f V \cos\Theta \qquad (44)$$

$$\frac{dh}{d\tau} = t_f \sin\Theta V \qquad (45)$$

$$\frac{dV}{d\tau} = t_f\left\{48.1965\pi - V^2\left[(.0000149 + .0009315\alpha^2)(1.-.0000069h)^{4.256}\right] - 32.131\sin\Theta\right\} \qquad (46)$$

$$\frac{d\Theta}{d\tau} = t_f\left\{\left(\frac{32.131}{V}\right)\left[1.5\pi\alpha + .000116V^2\alpha(1.0-.0000069h)^{4.256} - \cos\Theta\right]\right\}. \qquad (47)$$

Mathematical Description of the Control Variables

With the equations to be integrated constrained to the interval (0,1), the mathematical form chosen to describe the control variables is a series using Chebyshev polynomials, also defined on the interval (0,1). The Chebyshev polynomials, T_j, represented as functions of τ for j=1 to j=4 are:

$$T_1 = 1 \tag{48}$$

$$T_2 = 2\tau - 1 \tag{49}$$

$$T_3 = 8\tau^2 - 8\tau + 1 \tag{50}$$

$$T_4 = 32\tau^3 - 48\tau^2 + 18\tau - 1 . \tag{51}$$

The control variables are then described in series form by

$$\pi = \sum_{n=1}^{NPI} C_n T_n \tag{52}$$

$$\alpha = \sum_{m=1}^{NA} D_m T_m \tag{53}$$

where C and D are the unknown coefficients and part of the vector A. NPI and NA are the number of the unknown coefficients for π and α , respectively.

Constraints on the Control Variables

As discussed in earlier sections, physical aircraft characteristics limit the performance. These limitations are applied to the problem in the form of control variable inequality constraints. The thrust control constraint initially defined by equation (27) becomes

$$-.6 \leq \pi \leq 1.0 \tag{54}$$

The minimum value of -.6 represents an estimate of the maximum reverse thrust available in a turn.

Two separate constraints exist for the angle of attack. The dividing line between these two constraints is an altitude related parameter called corner velocity, V_c. Corner velocity is the velocity at which the lift coefficient required for flight at maximum load factor is equal to the maximum lift coefficient. Based on this definition and the related aircraft parameters, the corner velocity becomes

$$V_c = \left(\frac{62660.6}{\alpha_{MAX}\,\sigma}\right)^{1/2} \tag{55}$$

or

$$V_c = 559.735\,\sigma^{-1/2} \tag{56}$$

When the aircraft flies below corner velocity, the limiting constraint simply becomes the angle of attack limitation

$$\alpha \leq .2 \tag{57}$$

Above the corner velocity the aircraft is constrained by the structural load factor

$$\frac{\rho_0 \sigma V^2 S C_{L\alpha} \alpha}{2W} \leq \left(\frac{L}{W}\right)_{MAX} . \qquad (58)$$

The solution of equation (58) for α is

$$\alpha \leq \frac{62660.6}{\sigma V^2} \qquad (59)$$

when the known values of the parameters are incorporated. These two constraints as a function of velocity can be seen pictorally in Figure 1.

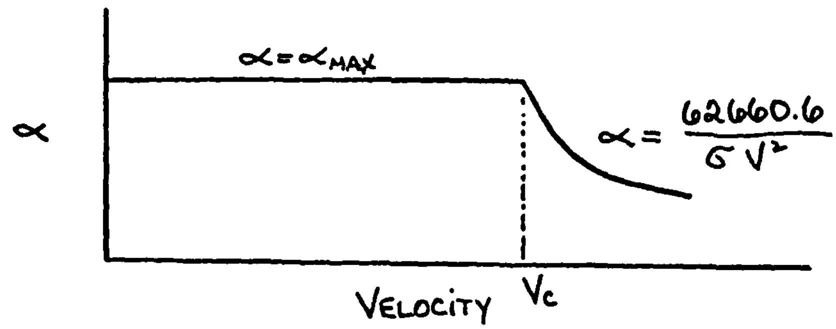

Figure 1. Angle of Attack vs. Velocity
(*Ref* (2))

The division on the angle of attack constraint suggests a method for determining which constraint applies. It is only required to monitor the value of the aircraft velocity throughout the integration process. The angle of attack calculation depends on whether the aircraft velocity is above or below the corner velocity for that altitude.

The method for thrust control is slightly more involved. When it is determined that the thrust control has at any time intersected its maximum or minimum value during integration, intermediate times, τ_1, τ_2, etc., must be introduced to allow the equations to be integrated from one point of boundary intersection to the next. This technique ensures that the thrust control never violates the constraints.

It is most important at this point to discuss the time during which the control variables have exceeded their allowable limitations for the entire trajectory. The action required during the time that the control variables have exceeded their constrained value is more easily explained by example. Consider the thrust control, if the thrust control (τ) is at its maximum value for the entire trajectory and if the F_A term corresponding to that control is negative, then in order to alter the coefficient so as to drive the corresponding F_A term to zero, it is required that the coefficient exceed its maximum allowable value. In similar fashion, if τ is at its minimum value and the corresponding F_A term is positive, then the τ coefficient must fall below its minimum allowable value. In either case, the control variable is no longer a parameter in the optimization process because its value is constant (upper or lower limit) and its corresponding F_A term cannot be driven to zero as desired. The fact that the F_A term for that control is not zero is no cause for concern. The minimum of the performance index with respect to that control occurs on the boundary and for a boundary minimum the F_A term is not supposed to be zero. The ultimate solution to this problem is to eliminate the coefficient (τ) in order to calculate $\delta\Omega$ and δA without its influence.

Numerical Derivative Computations

To compute $\delta\nu$ and δA, it is necessary to compute the derivative matrices M_A, M_{AA}, F_A, and F_{AA}. The latter two can be evaluated from the expressions:

$$F_A = G_A + \nu^T M_A \tag{60}$$

$$F_{AA} = G_{AA} + \nu M_{AA} \tag{61}$$

where G_A is known analytically (since $G=t_f$, $G_{t_f}=1$, and all other $G_A=0$) and $G_{AA}=0$. Therefore, the only unknowns requiring calculation are M_A and M_{AA}.

In order to calculate the end constraint value, M, and its derivatives M_A and M_{AA}, the equations of motion must be integrated to yield $X_f(A)$ from which M can be calculated directly. The M_A and M_{AA} matrices come from a central difference numerical technique. Let A_n represent the nominal value of the individual elements of A; then, the equations of motion are integrated to arrive at the nominal value of M. The first derivative, M_A, is determined by first perturbing A_n by some δ_n positively and then negatively,

$$A_{N+} = A_N + \delta_N \tag{62}$$

$$A_{N-} = A_N - \delta_N \tag{63}$$

where δ_n is some small positive value to be discussed later. Integration of the equations of motion using these perturbed values, A_{n+} and A_{n-}, results in M_+ and M_-, respectively. The M_{A_n} derivative is then calculated using the central difference representation

$$M_{A_N} = \frac{M_+ - M_-}{2\delta_N} + \mathcal{O}(\delta_N^2) \qquad (64)$$

where $\mathcal{O}(\delta_n^2)$ represents an error term of order of magnitude δ_n^2. The M_{AA} matrix is determined similarly; however, two elements A_n and A_m must be perturbed both positively and negatively to yield M_{++}, M_{--}, M_{+-}, and M_{-+} after integration. The central difference representation for the M_{AA} matrix are

$$M_{A_N A_M} = \frac{M_+ - 2M + M_-}{\delta_N^2} + \mathcal{O}(\delta_N^2) \qquad (65)$$

if n=m

$$M_{A_N A_M} = \frac{M_{++} - M_{+-} - M_{-+} + M_{--}}{4\delta_N \delta_M} + \mathcal{O}(\delta_N \delta_M) \qquad (66)$$

if n≠m. The greatest accuracy in the central difference method is obtained by using the smallest δ_n and δ_m possible; if δ_n and δ_m are small enough, the error terms can be ignored. The selection is a critical matter in the integration process. Care must be taken to ensure that the numerators of equations (64) thru (66) are not of the same order of magnitude as the truncation error associated with integration. The δ chosen for the central difference technique is

$$\delta_N = (\text{DELTA})\, A_N \,, \tag{67}$$

or if the absolute value of δ_n is smaller than DELTA , then

$$\delta_N = \text{DELTA} \tag{68}$$

where DELTA is a given small positive number. Therefore, δ_n is controlled by the value of DELTA. This study found that DELTA equal to 10^{-4} was an appropriate value although occasional manipulation to 10^{-3} and 10^{-2} was required to stabilize the optimization process.

Convergence Criteria

The scaling factors P and Q control convergence of the iterative process. In order for the method to ultimately converge, both P and Q must equal one. During this study it was found that Q could be left equal to one throughout the iterative process. The value of P started at 10^{-4} and increased by a factor of 10 as long as the norm of F_A^T was continuously decreasing for the previous value of P. Once P equals one, the convergence criteria becomes:

$$\|M\| \le 10^{-8} \tag{69}$$

$$\|\delta A\| \le 10^{-4} \tag{70}$$

Initializing Control Variables

With the aircraft trajectories constrained to the vertical plane, the bank angle control is removed from the problem, leaving only angle of attack and throttle as the maneuver controls. Since lift turns the aircraft, the initial angle of attack is the maximum value as described in Figure 1. The initial throttle control depends on whether the maneuver is a pull-up or split-s, and whether the initial velocity is above or below corner velocity. Since the quickest turns will be made at corner velocity, the general philosophy used in selecting the initial throttle control is to choose that throttle which will bring the aircraft to corner velocity and hold it there throughout the turn. The set of maneuvers are initiated with a constant throttle control. Once each case has converged on a solution, an additional coefficient is added to the control polynomial until there is no improvement in the turning time. Each of the four maneuvers is handled in the same way. The four maneuvers in this study are the pull-up high velocity), the pull-up (low velocity), the split-s (high velocity), and the split-s (low velocity), where high velocity refers to initial velocity above corner velocity, and low velocity refers to initial velocity below corner velocity.

V. Results of the Study

Individual Maneuvers

The results of the throttle control for each of the four maneuvers are covered separately in the following paragraphs. The results of angle of attack control are discussed first because the results were identical for all four maneuvers. As expected, the aircraft generates as much lift as possible for minimum time turns in the vertical plane. In order to generate this lift, the aircraft remains at its maximum angle of attack limit throughout each of the four maneuvers for all throttle excursions. This limit is .2 radians for velocities below corner velocity and the angle of attack for maximum load factor for velocities above corner velocity. The interesting point of the angle of attack optimization process is that the corresponding F_A term was found to be negative in all cases indicating that the coefficient desired to be greater than its maximum allowable value.

The results for the low velocity split-s maneuver are listed in Table 2 and shown in Figures 2 thru 5. Table 2 lists the optimal coefficients for the various throttle control polynomials and the corresponding Lagrange multipliers. From Table 2 it can be seen that the best turning times are obtained for throttle controls of quadratic and cubic order. The improvement in the turning time with cubic throttle control over quadratic control is negligible (less than one tenth of one percent). The quadratic control shows an improvement in turning time of 3% and 4% over linear and constant throttle controls, respectively. Figures 2a, 3a, 4a, and 5a show the altitude-velocity profiles for constant,

	A	Constant	Linear	Quadratic	Cubic
Minimum Time	t_f	.9631653E+01	.9554657E+01	.9278816E+01	.9271625E+01
Bank Angle Coefficient	B_1	0.0	0.0	0.0	0.0
Thrust Control Coefficients	C_1	-.5792143E-01	-.9792333E-01	.3106268E+00	.2138034E+00
	C_2		-.3417159E+00	-.1713567E+00	-.5497406E+00
	C_3			.8249045E+00	.7008462E+00
	C_4				-.2250979E+00
Angle of Attack Coefficient	D_1	0.2 if $V<V_c$, $\alpha=(62660.6/\sigma V^2)$ if $V>V_c$			
Lagrange Multiplier	ν	-.3267441E+01	-.3496762E+01	-.3248849E+01	-.3152788E+01

Table 2. Optimal Coefficients for the Split-S, Low Velocity Maneuver

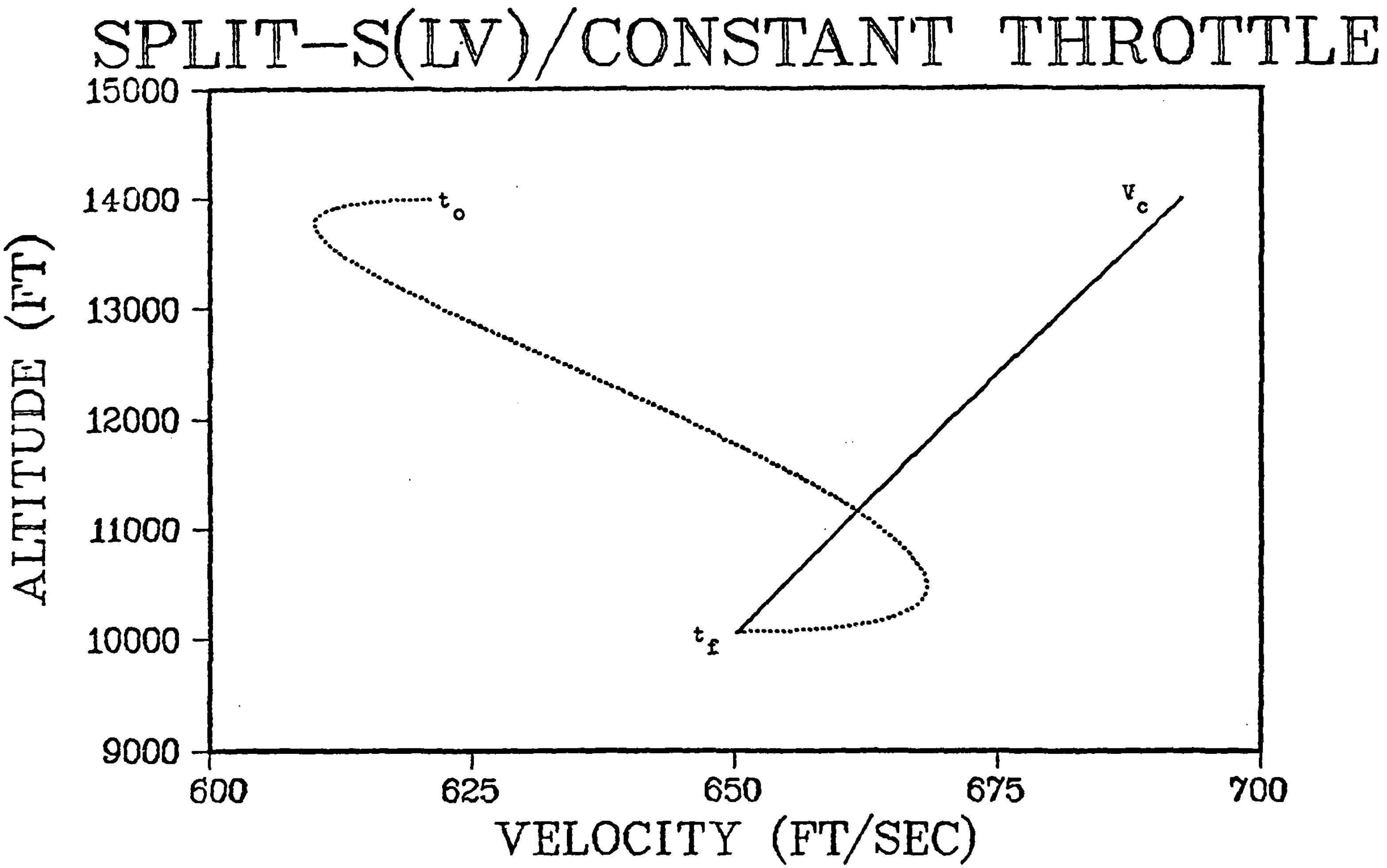

Figure 2a. Altitude vs. Velocity

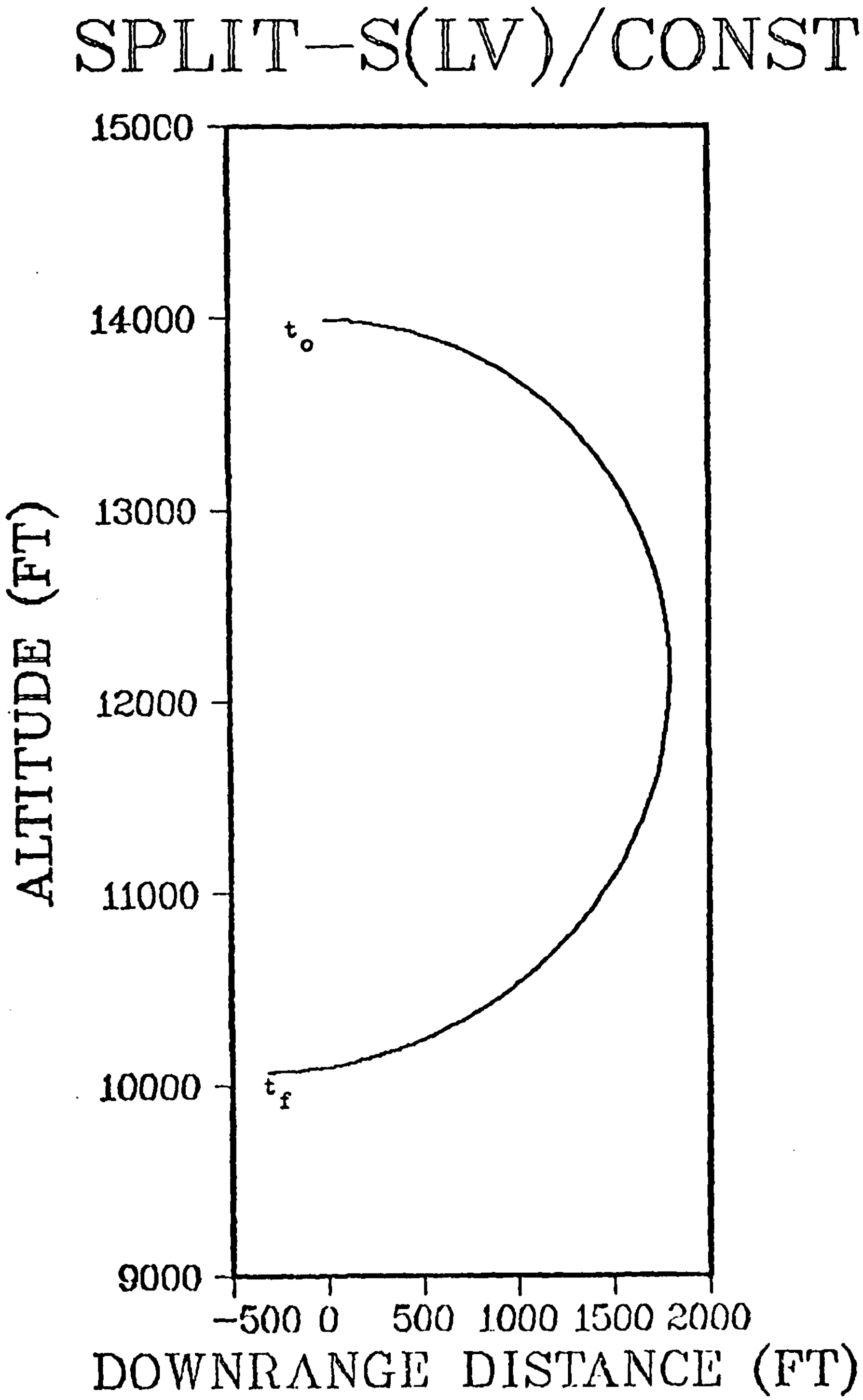

Figure 2b. Altitude vs. Downrange Distance

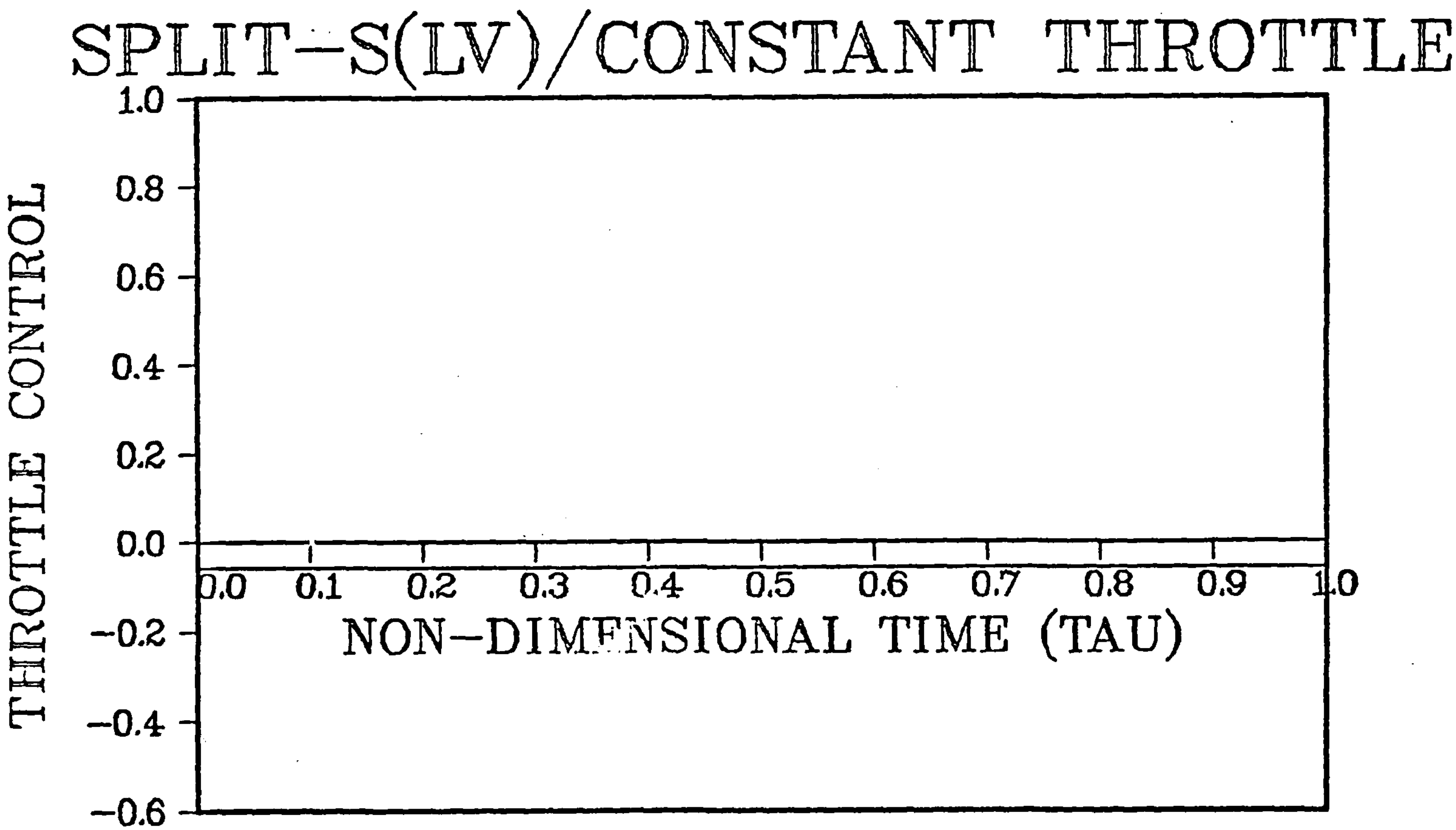

Figure 2c. Throttle Control vs. Non-Dimensional Time

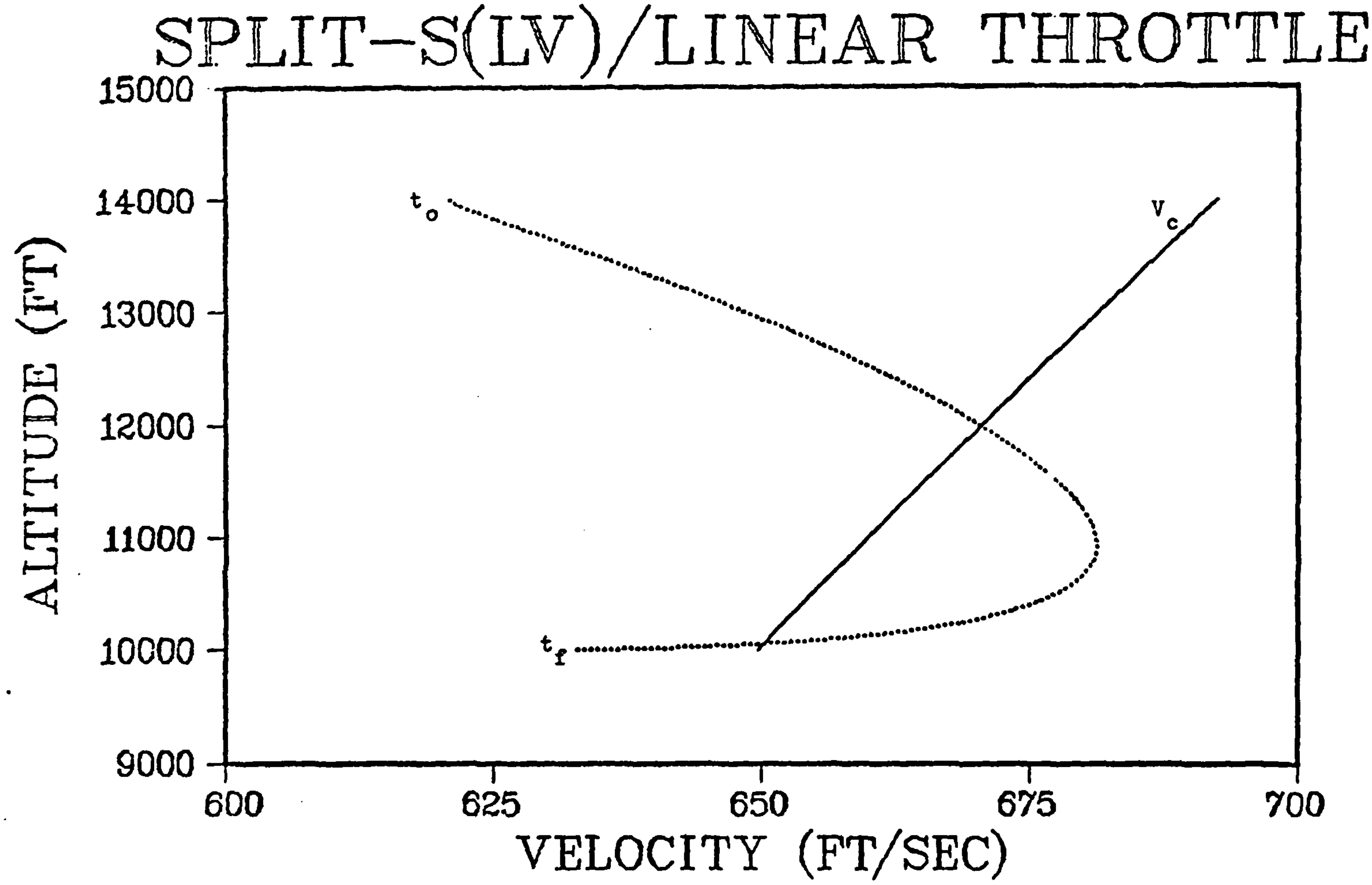

Figure 3a. Altitude vs Velocity

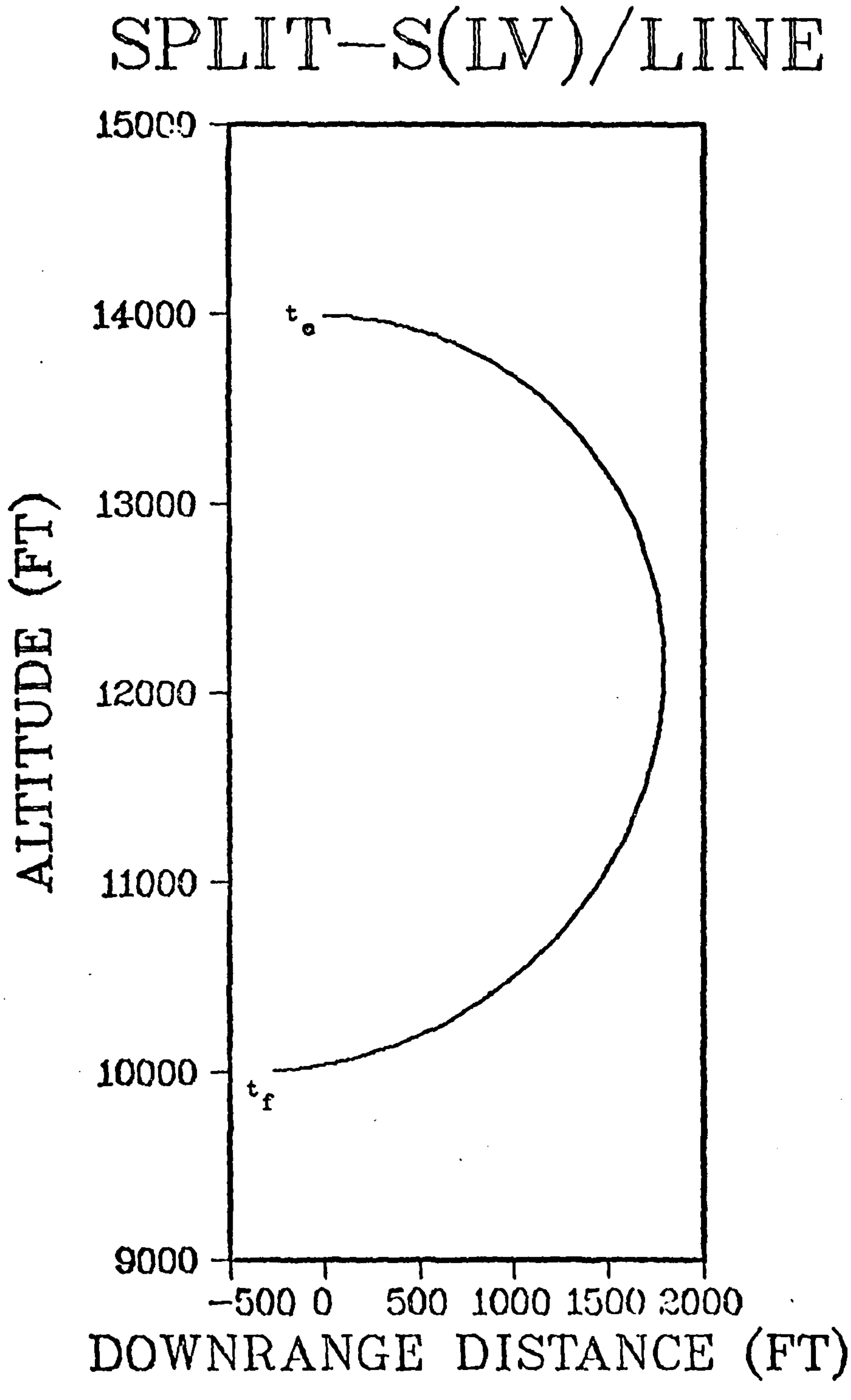

Figure 3b. Altitude vs. Downrange Distance

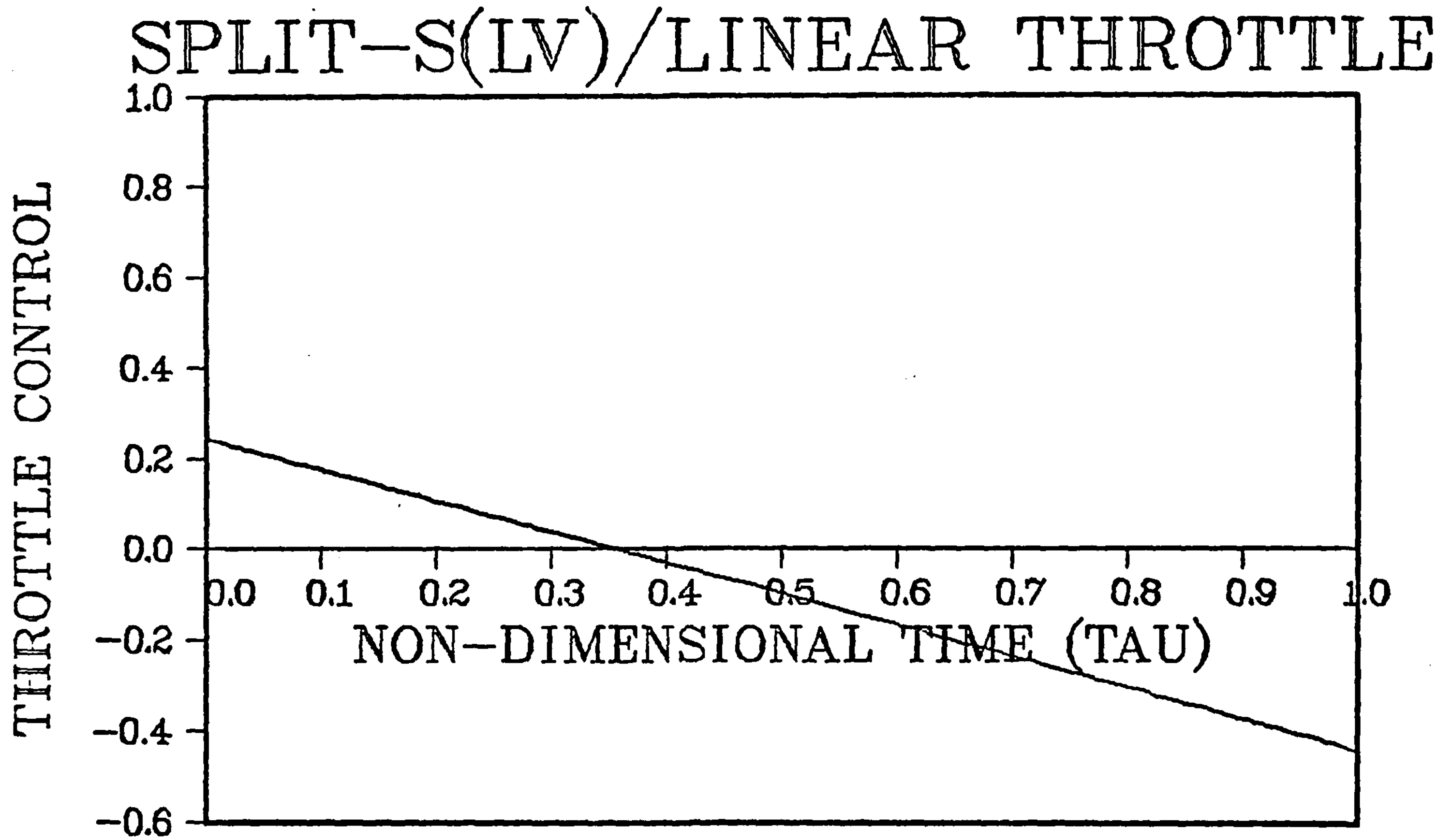

Figure 3c. Throttle Control vs. Non-Dimensional Time

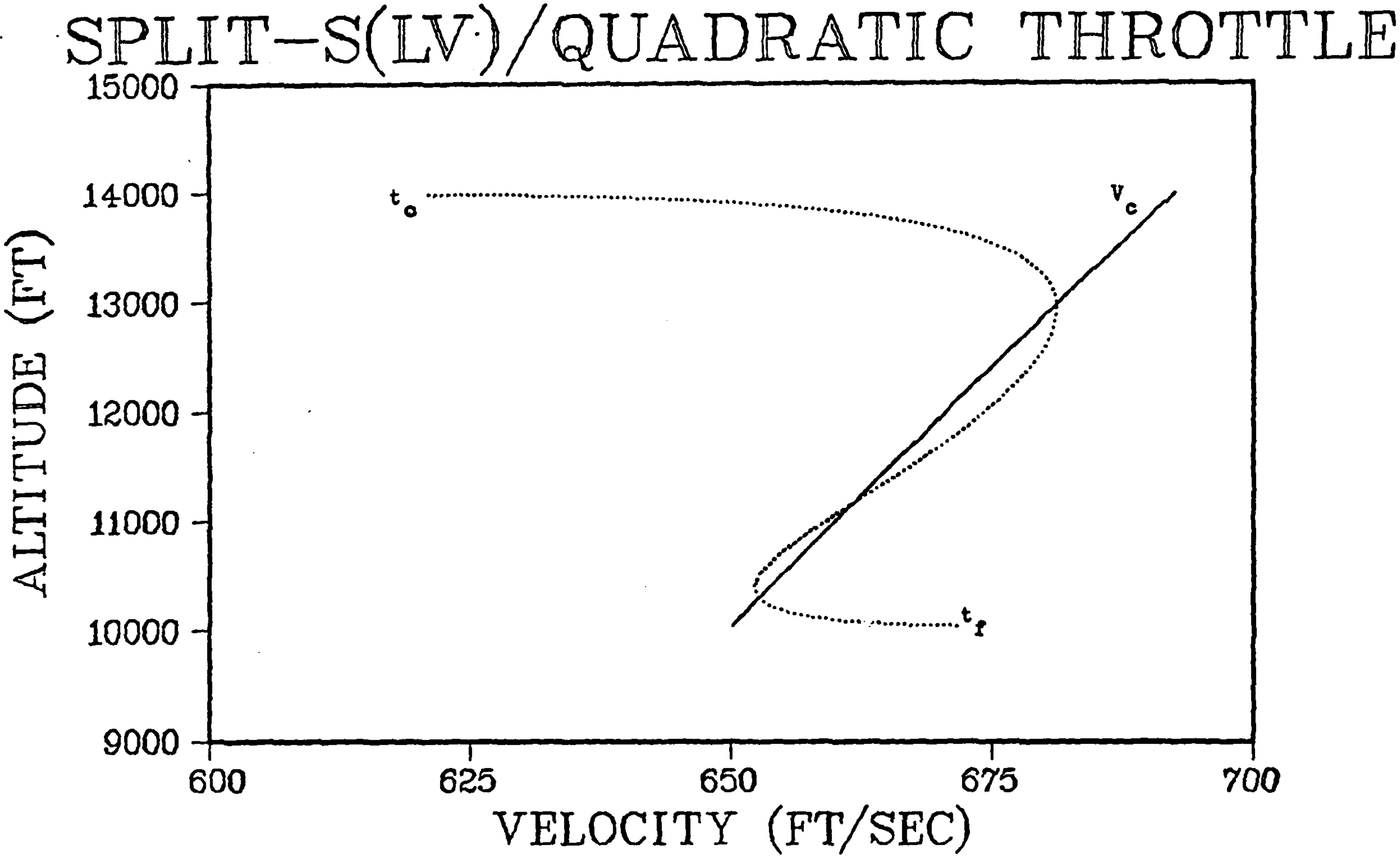

Figure 4a. Altitude vs. Velocity

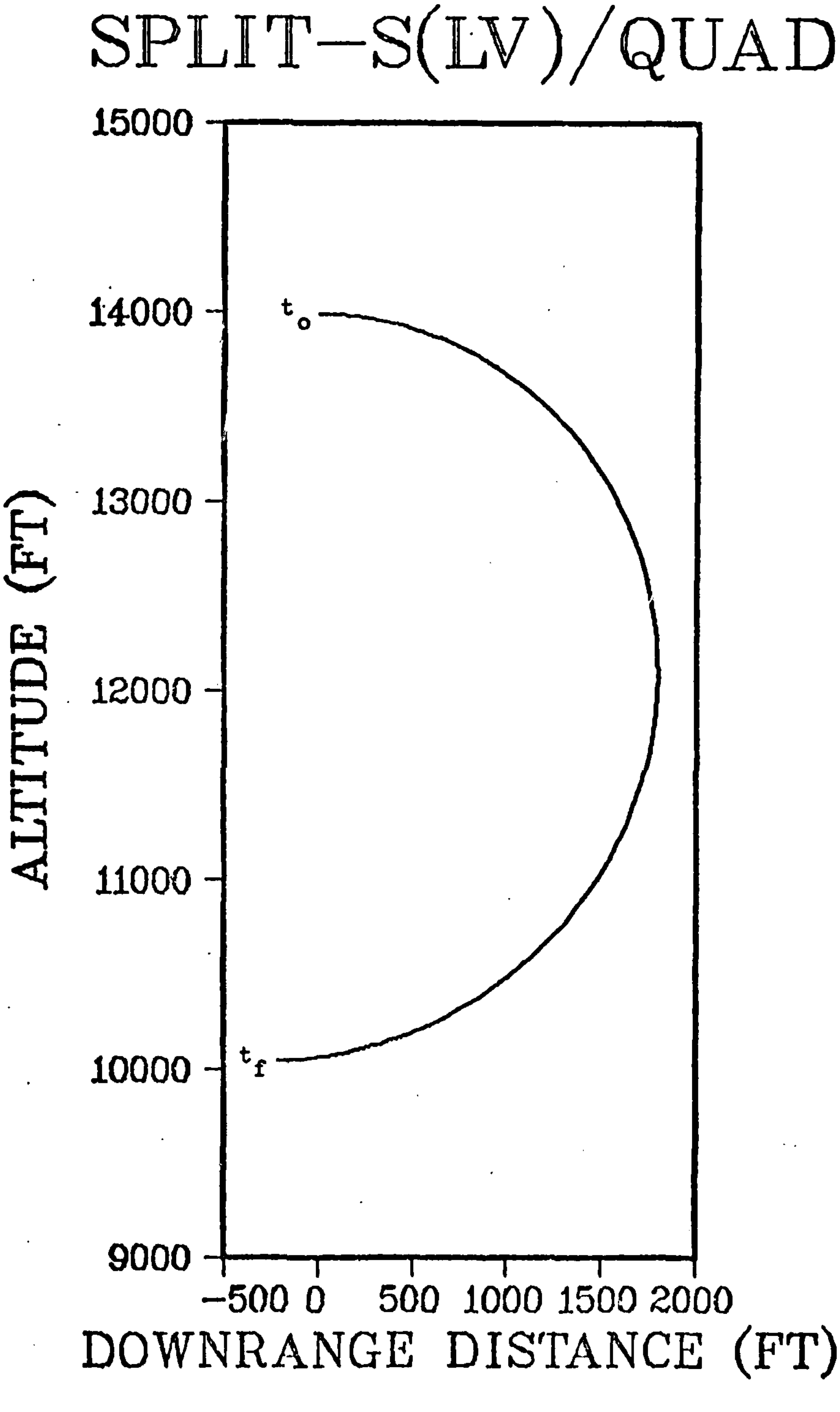

Figure 4b. Altitude vs. Downrange Distance

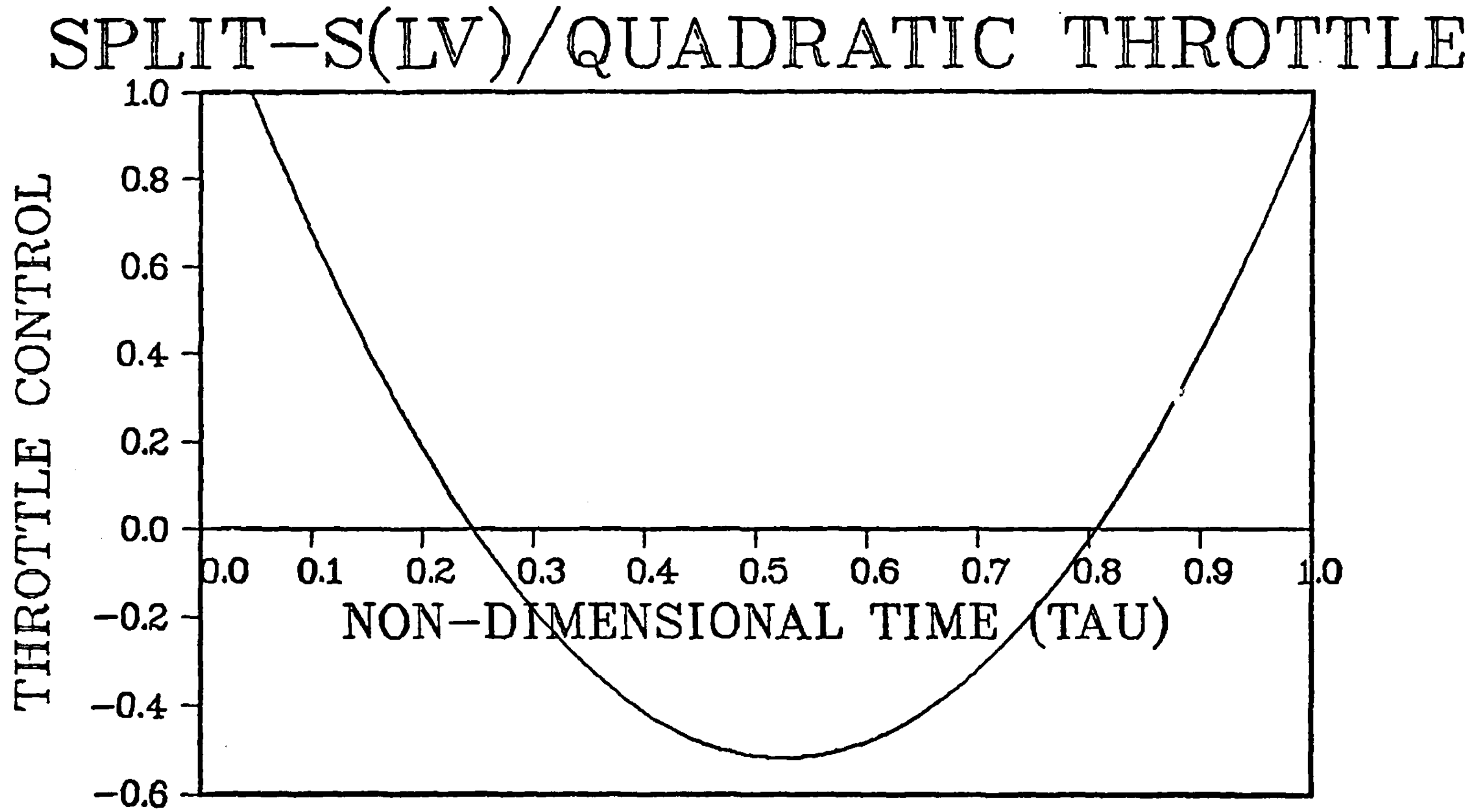

Figure 4c. Throttle Control vs. Non-Dimensional Time

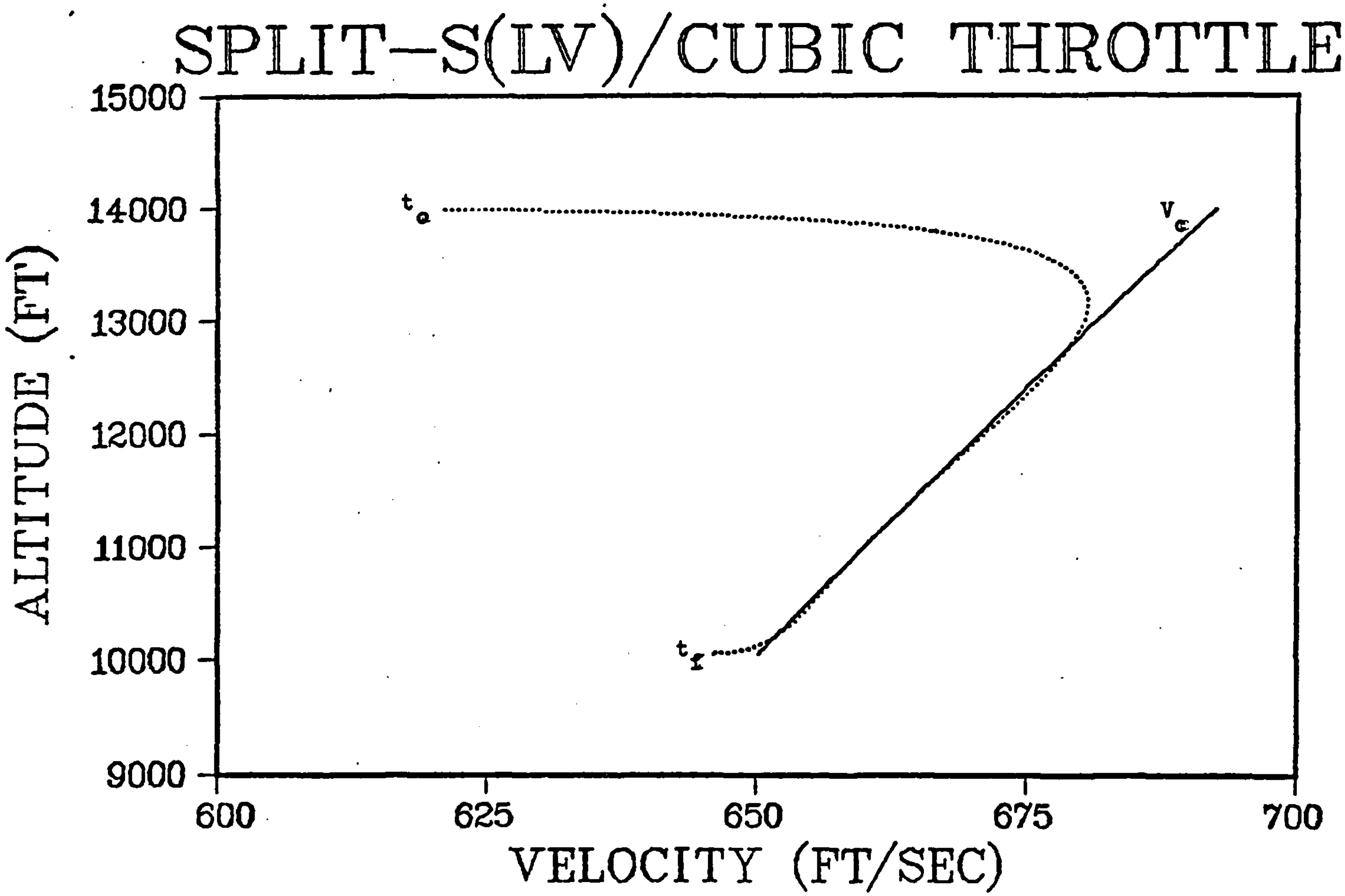

Figure 5a. Altitude vs. Velocity

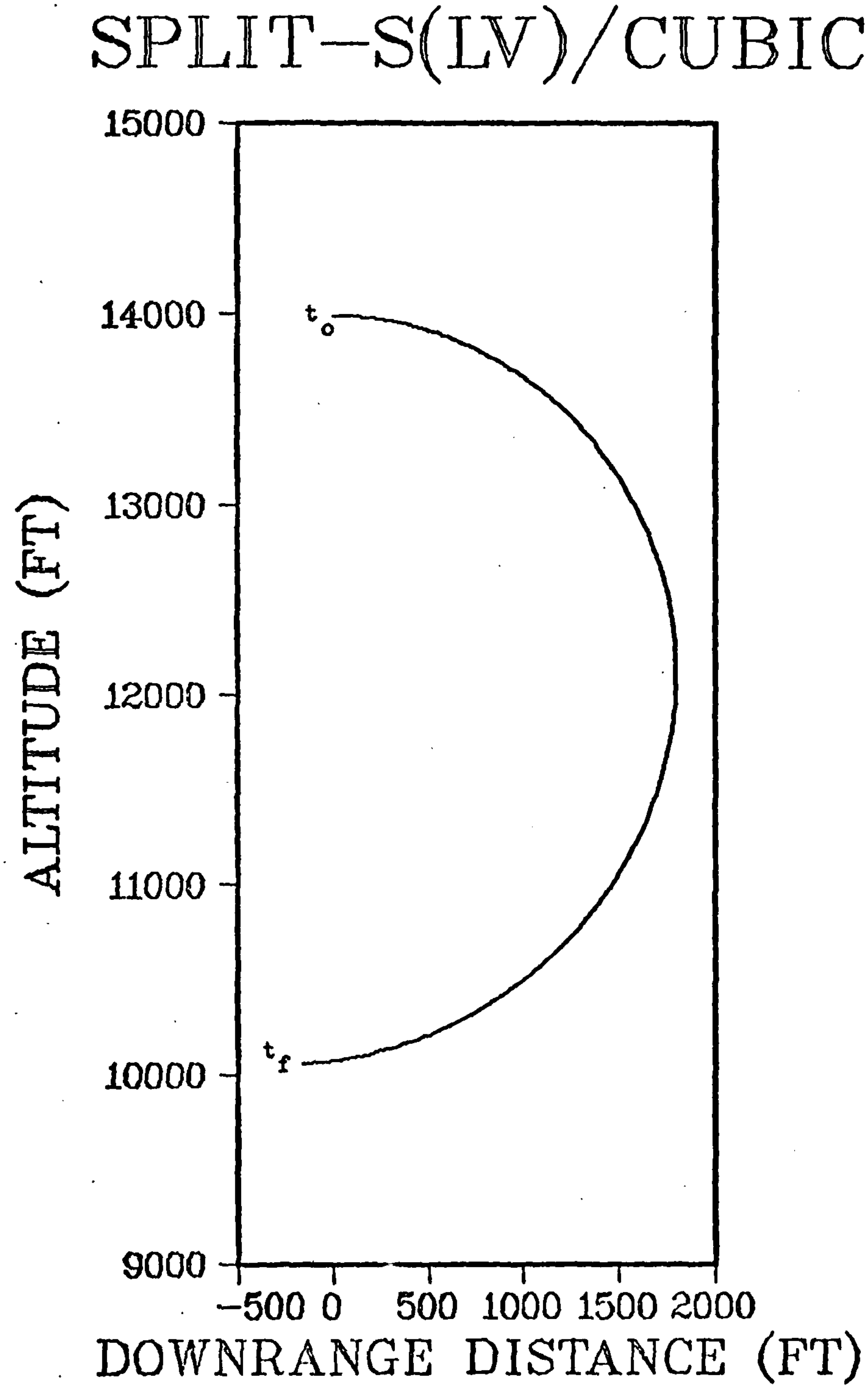

Figure 5b. Altitude vs. Downrange Distance

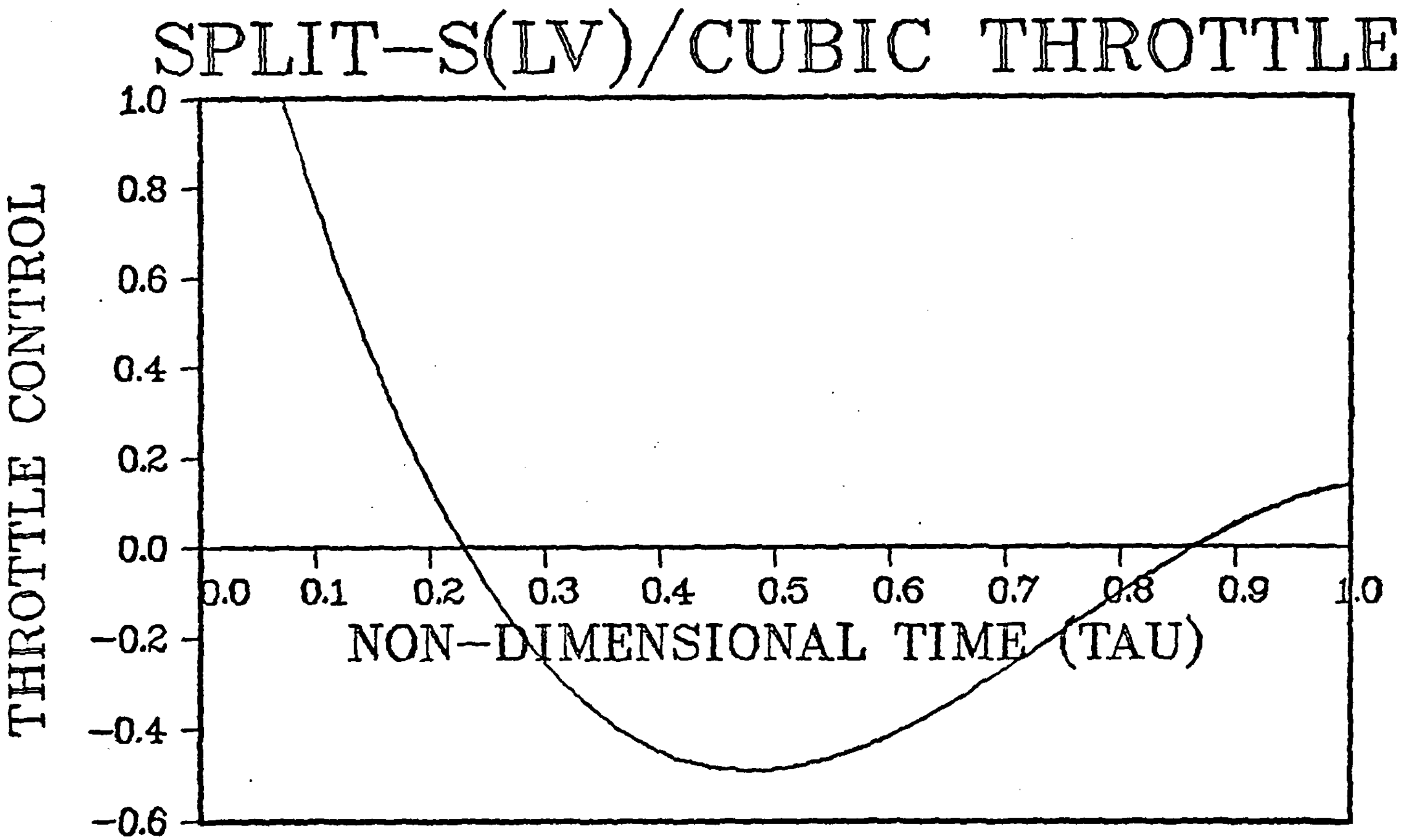

Figure 5c. Throttle Control vs. Non-Dimensional Time

linear, quadratic, and cubic throttles, respectively. In each of the figures, the dotted line originating at t_o and ending at t_f is the aircraft trajectory, while the solid line labeled V_c is the corner velocity as a function of altitude. As stated earlier in the text, the throttle control which can bring the aircraft to corner velocity as quickly as possible and hold it there will give the minimum turning times. The truth of this statement is readily apparent in Figures 2a, 3a, 4a, and 5a. As the order of the throttle control increases the aircraft's ability to maintain corner velocity improves noticeably and the time to turn decreases appropriately. The maneuver in the x-h plane for each of the four throttle control polynomials appear in Figures 2b, 3b, 4b, and 5b. The trajectory in each figure begins at t_o and is completed at t_f. Each of the four trajectories are bacically identical in the x-h plane. There are minimal differences in the final altitude and downrange distances. The difference in final altitude among the four throttle controls is only 68 ft with the minimum being 10,001 ft for the linear throttle and the maximum being 10,069 ft for the constant throttle case. Final altitudes for quadratic and cubic throttle controls fell between these two values. Comparison of final downrange distances show that the values ranged from -289 ft for the cubic throttle case to -390 ft for the linear throttle case while constant and quadratic throttle controls were -335 ft and -301 ft, respectively. The aircraft throttle controls which lead to the trajectories for the low velocity split-s maneuver are shown in Figures 2c, 3c, 4c, and 5c. Figure 2c shows that a small value of constant throttle (π =-.0579) was required to perform the maneuver in minimum time. For the linear throttle control case (Figure 3c), it was found that a small positive initial throttle control, $\pi = .24$, was required to drive the aircraft toward corner velocity. The throttle then decreased linearly to a minimum value at approximately -.4 at the maneuver's completion. For the quadratic and cubic throttle controls, Figures 4c and 5c, the throttle controls were very similar with the cubic throttle control dropping off at the maneuver's completion. Conclusions regarding the best throttle control for this maneuver will be made in the Summary of Results.

The split-s (high velocity) maneuver began its trajectory with an initial velocity of 903 fps which is well above corner velocity of 694 fps. Table 3 lists the optimal coefficients for the split-s (high velocity) maneuver while Figure 6 displays the results. It was found that with the combination of high initial velocity and the split-s maneuver the aircraft was unable to decelerate to corner velocity even with full reverse throttle, $\pi = -.6$. This trajectory is displayed in Figure 6a. Figure 6b shows the trajectory in the x-h plane. This figure shows that while the high velocity split-s maneuver loses over 1300 ft more in altitude than the worst low velocity split-s case, it finishes the maneuver over 600 ft farther downrange. The throttle control for this maneuver is displayed in Figure 6c. Table 3 does not list a value for the Lagrange multiplier for this maneuver. The reason for this is that the aircraft was physically constrained to both its maximum angle of attack limit and its minimum throttle control constraint. Due to this, the problem was reduced simply to one of calculation of the

	A	Constant	Linear	Quadratic	Cubic
Minimum Time	t_f	.1078282E+02			
Bank Angle Coefficient	B_1	0.0			
Thrust Control Coefficient	C_1	-.6000000E+00			
	C_2				
	C_3				
	C_4				
Angle of Attack Coefficient	D_1	0.2 ir $V<V_c$, $\alpha=(62660.6/\sigma V^2)$ if $V>V_c$			
Lagrange Multiplier	ν				

Table 3. Optimal Coefficients for the Spit-S, High Velocity Maneuver

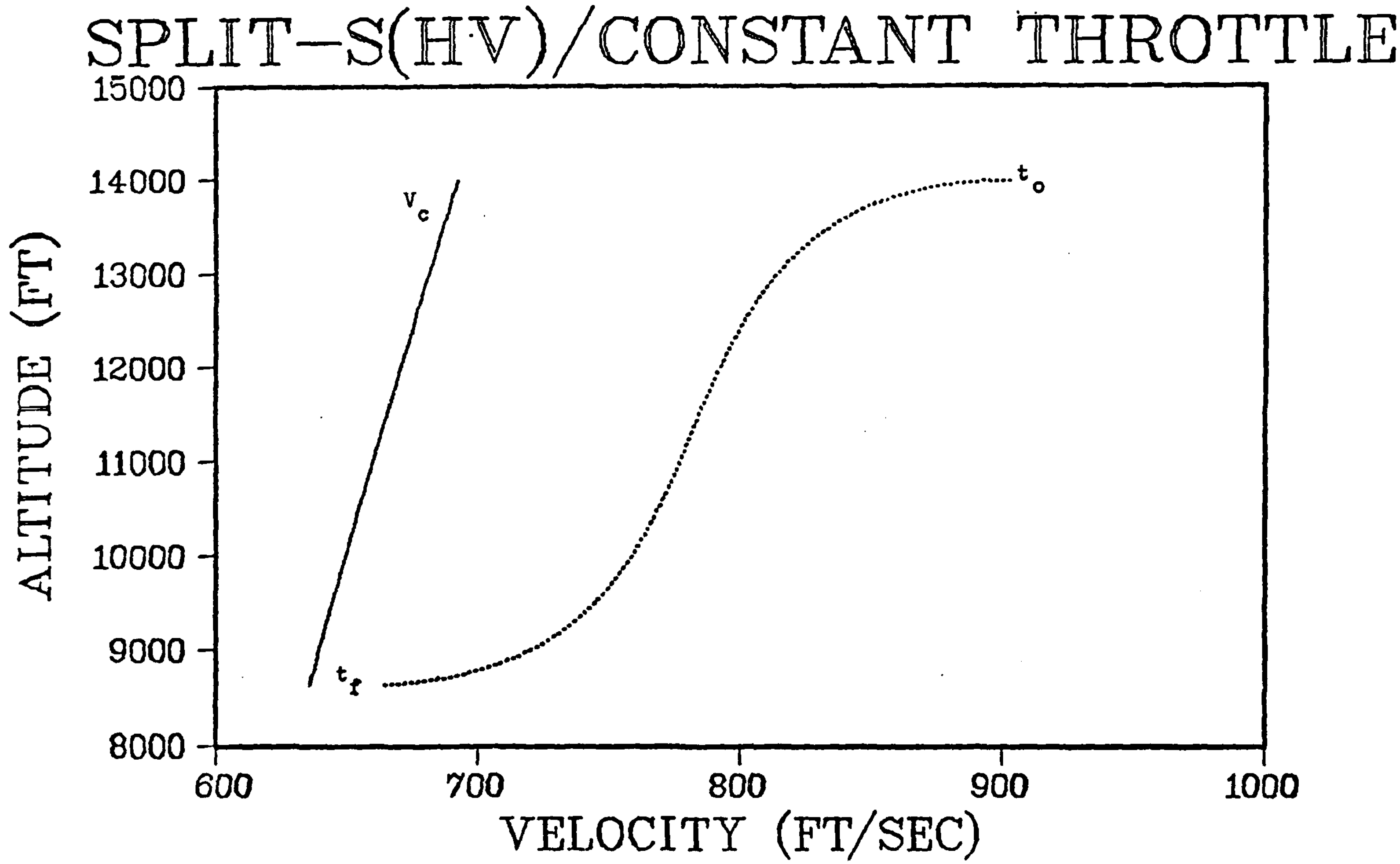

Figure 6a. Altitude vs. Velocity

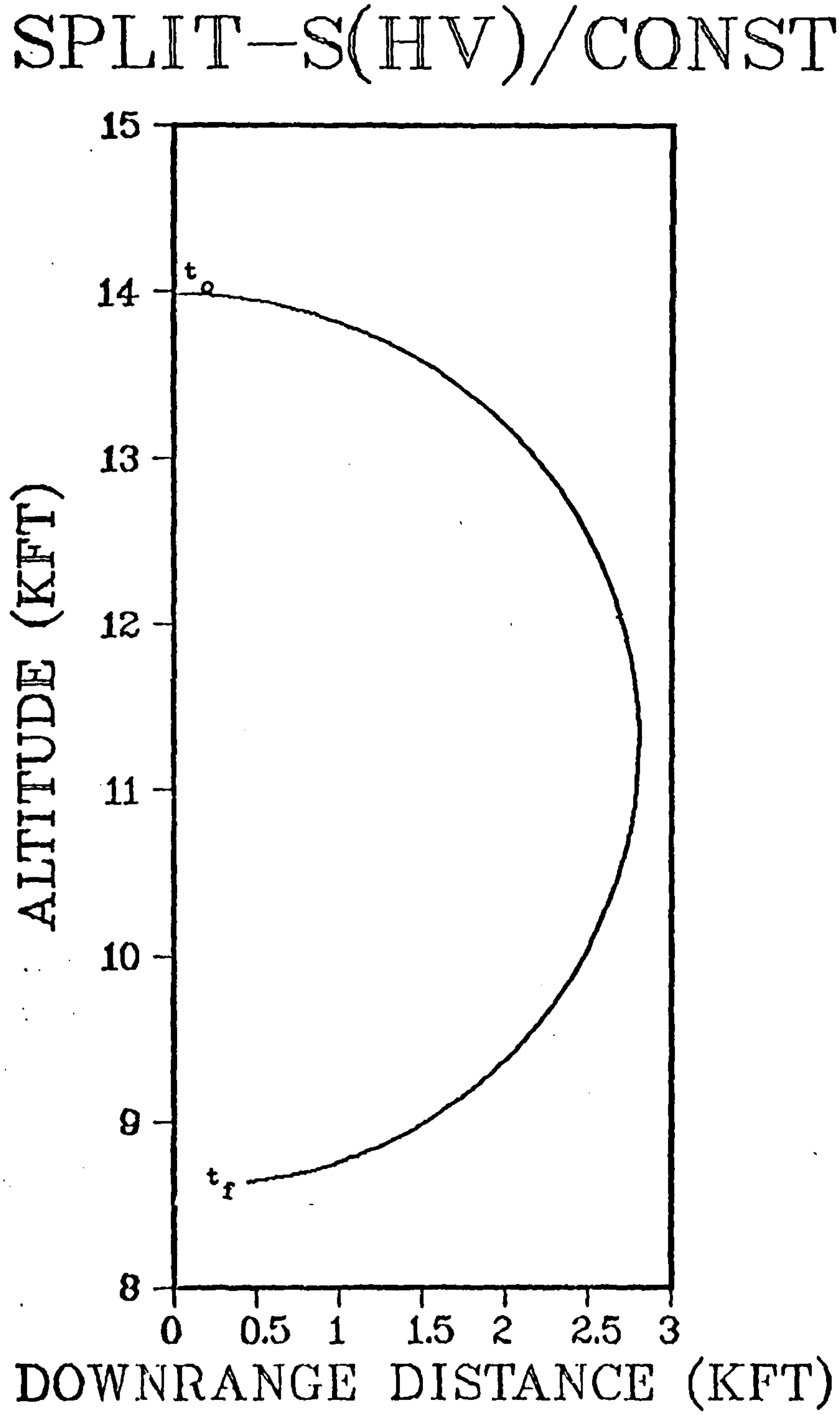

Figure 6b. Altitude vs. Downrange Distance

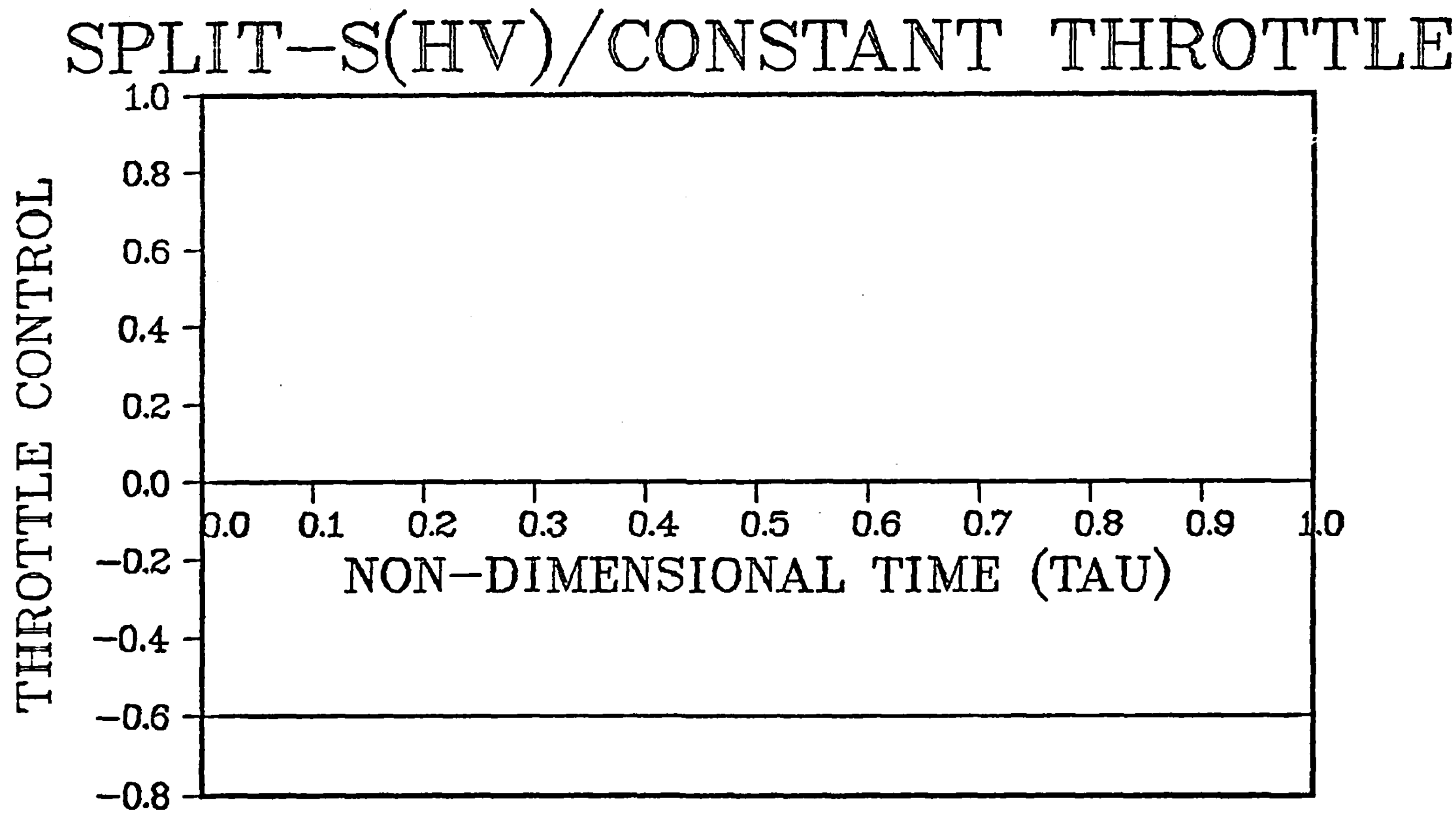

Figure 6c. Throttle Control vs. Non-Dimensional Time

end condition with constant aircraft controls.

For the low velocity pull-up maneuver, the optimal coefficients are listed in Table 4. It was found that the minimum time for this maneuver, t_f=9.7763 sec, results from a constant throttle control of π =1.0. The constant throttle control allowed the aircraft to accelerate vertically in the maneuver, however it did not attain corner velocity until the maneuver was completed. Figure 7 shows the trajectories and throttle control for the low velocity pull-up maneuver. From Figure 7a it is concluded that even though the aircraft has the capability of accelerating in the pull-up maneuver, the low initial velocity and the effect of gravity do not allow it to accelerate to corner velocity. Figure 7b displays the pull-up (low velocity) maneuver in the x-h plane. The figure shows that during the maneuver the aircraft gained over 4200 ft of altitude and completed the maneuver 249 ft downrange. Figure 7c shows the throttle control required for the maneuver.

The high velocity pull-up maneuver is shown in Figures 8 and 9. The optimal coefficients are listed in Table 5. Beginning the pull-up maneuver with an initial velocity much greater than corner velocity allowed for a number of interesting throttle control solutions. Figures 8a and 9a show the altitude-velocity profiles for the constant and linear throttle controls, respectively. It was found that for the constant throttle case a minimum turning time of 11.15 sec was obtained with a small positive throttle control, π =.3246, as shown in Figure 8c. When the throttle control was allowed to go to a linear control some in-

	A	Constant	Linear	Quadratic	Cubic
Minimum Time	t_f	.9776376E+01			
Bank Angle Coefficient	B_1	0.0			
Thrust Control Coefficients	C_1	.1000000E+01			
	C_2				
	C_3				
	C_4				
Angle of Attack Coefficients	D_1	0.2 if $V<V_c$, $\alpha=(62660.6/\sigma V^2)$ if $V>V_c$			
Lagrange Multiplier	ν	-.2729329E+01			

Table 4. Optimal Coefficients for the Pull-Up, Low Velocity Maneuver

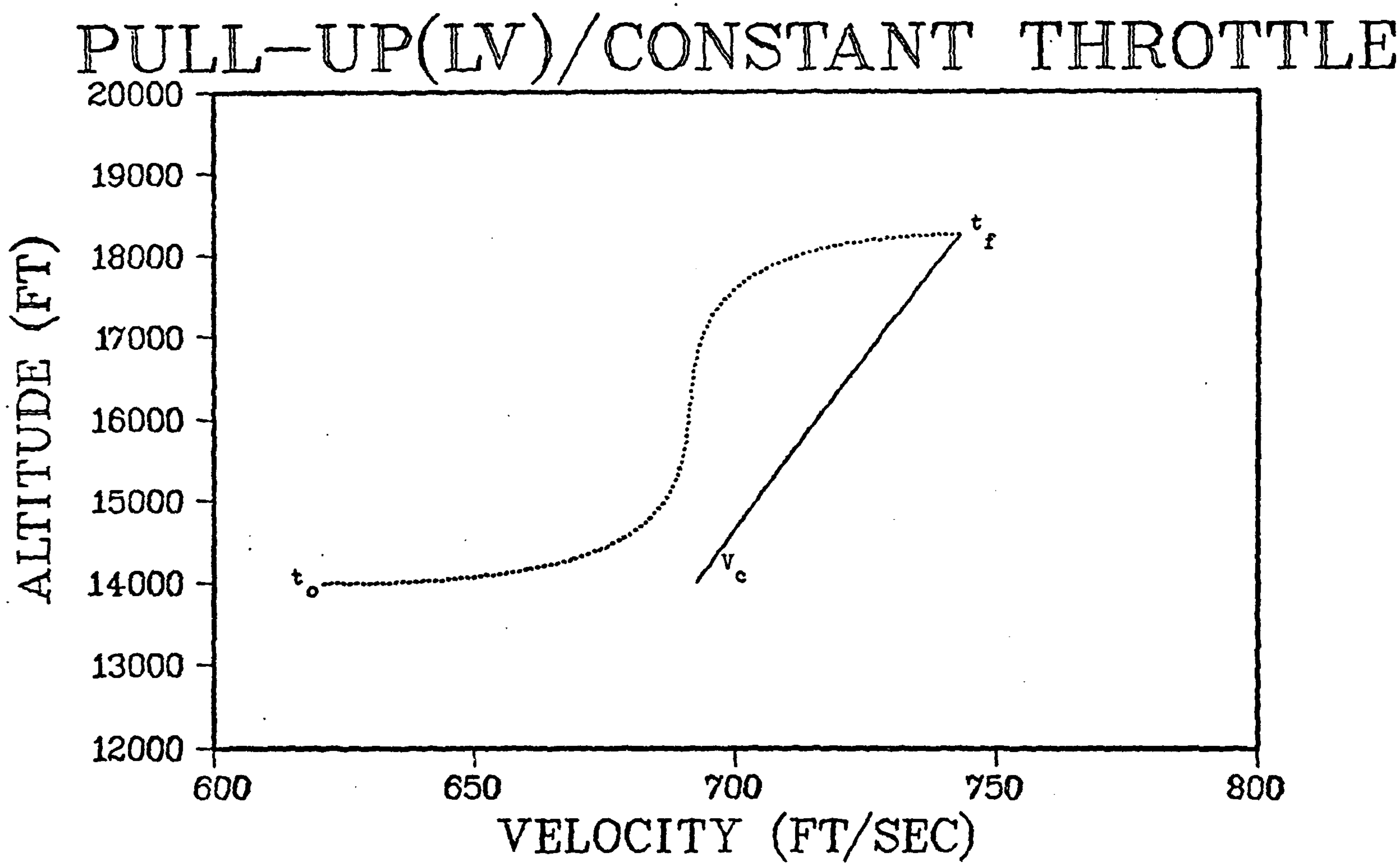

Figure 7a. Altitude vs. Velocity

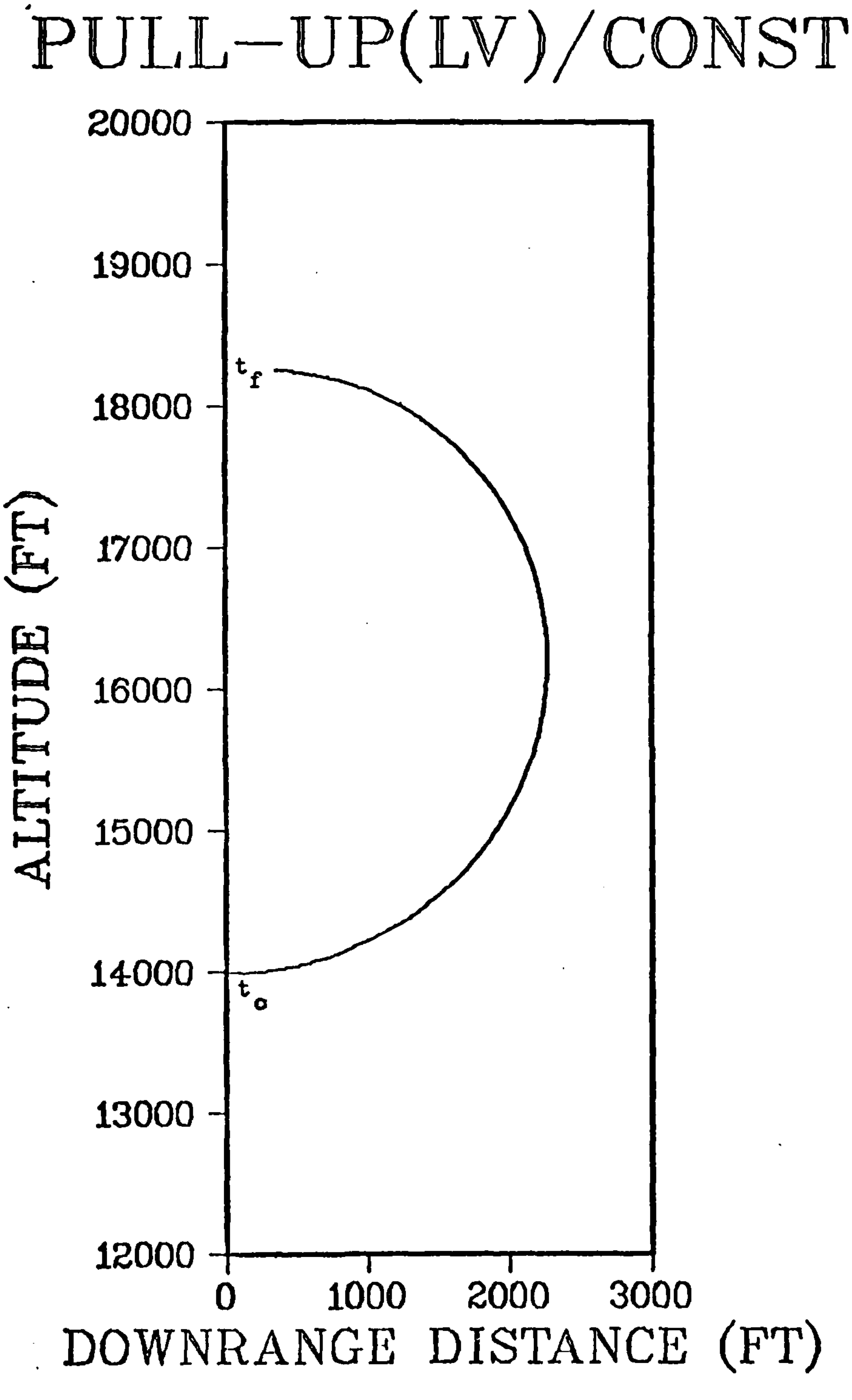

Figure 7b. Altitude vs. Downrange Distance

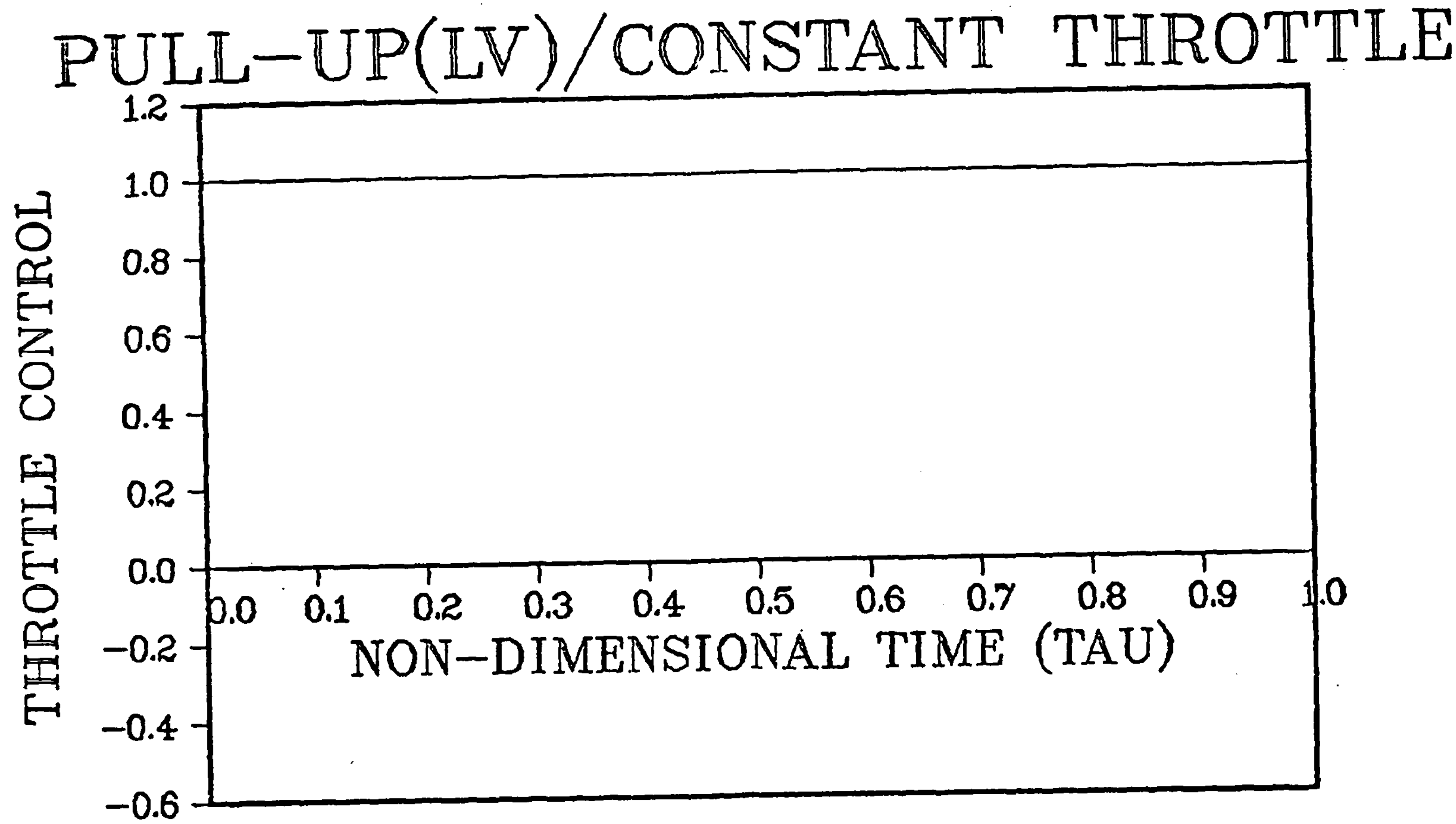

Figure 7c. Throttle Control vs. Non-Dimensional Time

teresting results were obtained. As the optimization process proceeded, the throttle began to converge on a solution that went from full reverse throttle to full forward throttle with infinite slope. This is a bang-bang control and the only parameter to describe it is the time at which the control switches from full reverse to full forward throttle control. The results of this control are shown in Figure 9. Figure 9a displays the altitude-velocity profile which results in a time to turn of 10.171 sec, a 9% improvement over the constant throttle case. The optimal time to switch from reverse to forward throttle was found to be 3.3923 sec. This throttle control is shown in Figure 9c. Figures 8b and 9b compare the x-h plane trajectories for the two throttle controls and it is seen that the linear throttle control completes the pull-up maneuver in over 900 ft less altitude and over 500 ft less downrange distance than the constant throttle control. When the optimization process was allowed to go on to higher order throttle controls it once again converges to the bang-bang control.

Each of the four maneuvers has been discussed separately. The following section will make some general comparisons between each of these maneuvers.

Summary of Results

In Table 6 all of the individual maneuvers performed in this study as well as the data from previous work (Table 1) are listed for comparative purposes. The minimum time to turn as well as aircraft specific energy will be discussed in this section. Specific energy can be calculated from the expression

	A	Constant	Linear	Quadratic	Cubic
Minimum Time	t_f	.1115269E+02	.1017108E+02		
Bank Angle Coefficient	B_1	0.0	0.0		
Thrust Control Coefficients	C_1	.3246432E+00	-.6000 if $t<3.3923$ sec; 1.000 if $t>3.3923$ sec		
	C_2				
	C_3				
	C_4				
Angle of Attack Coefficients	D_1	0.2 if $V<V_c$, $\alpha=(62660.6/\sigma V^2)$ if $V>V_c$			
Lagrange Multiplier	ν	-.3041485E+01	-.2820330E+01		

Table 5. Optimal Coefficients for the Pull-Up, High Velocity Maneuver

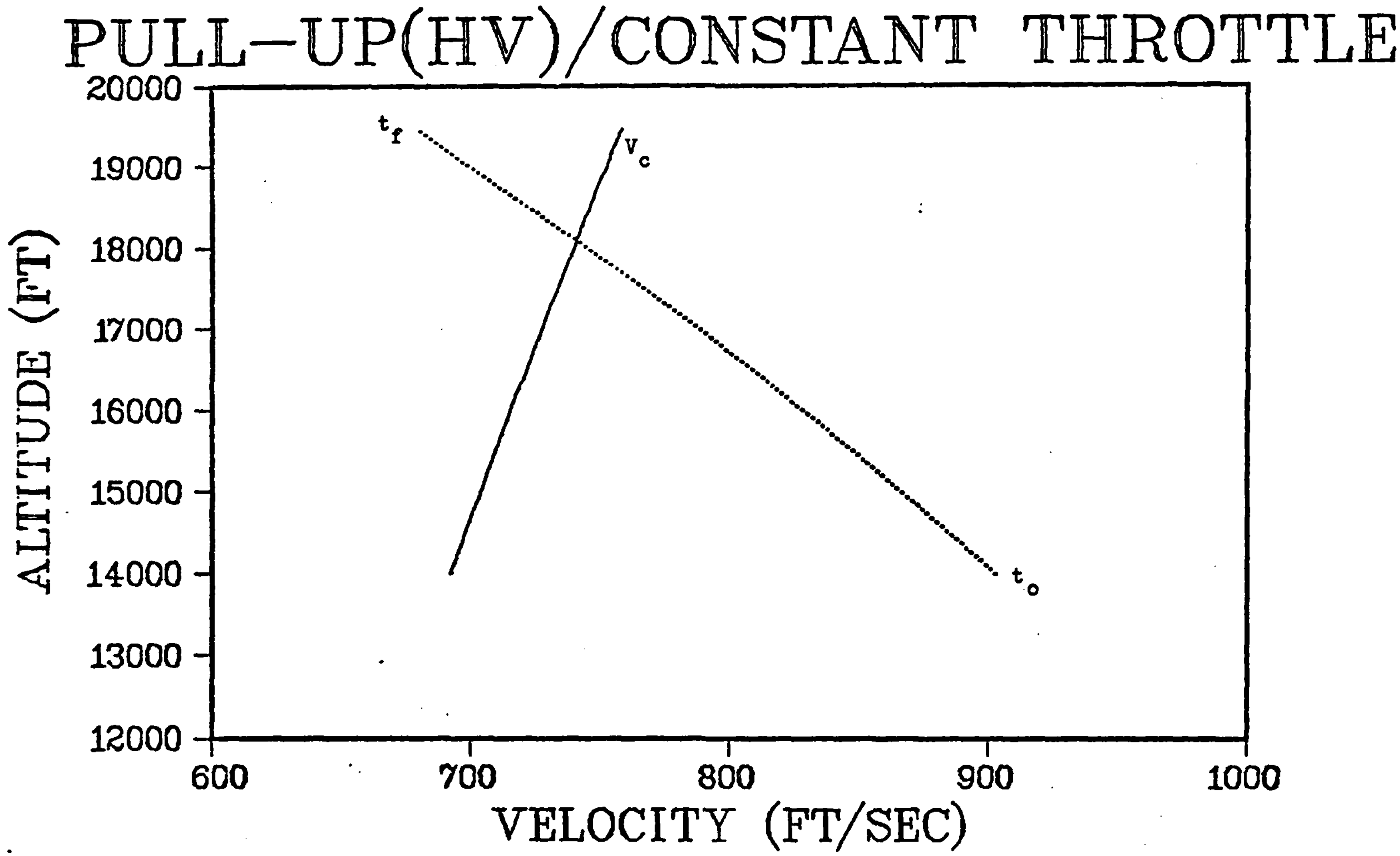

Figure 8a. Altitude vs. Velocity

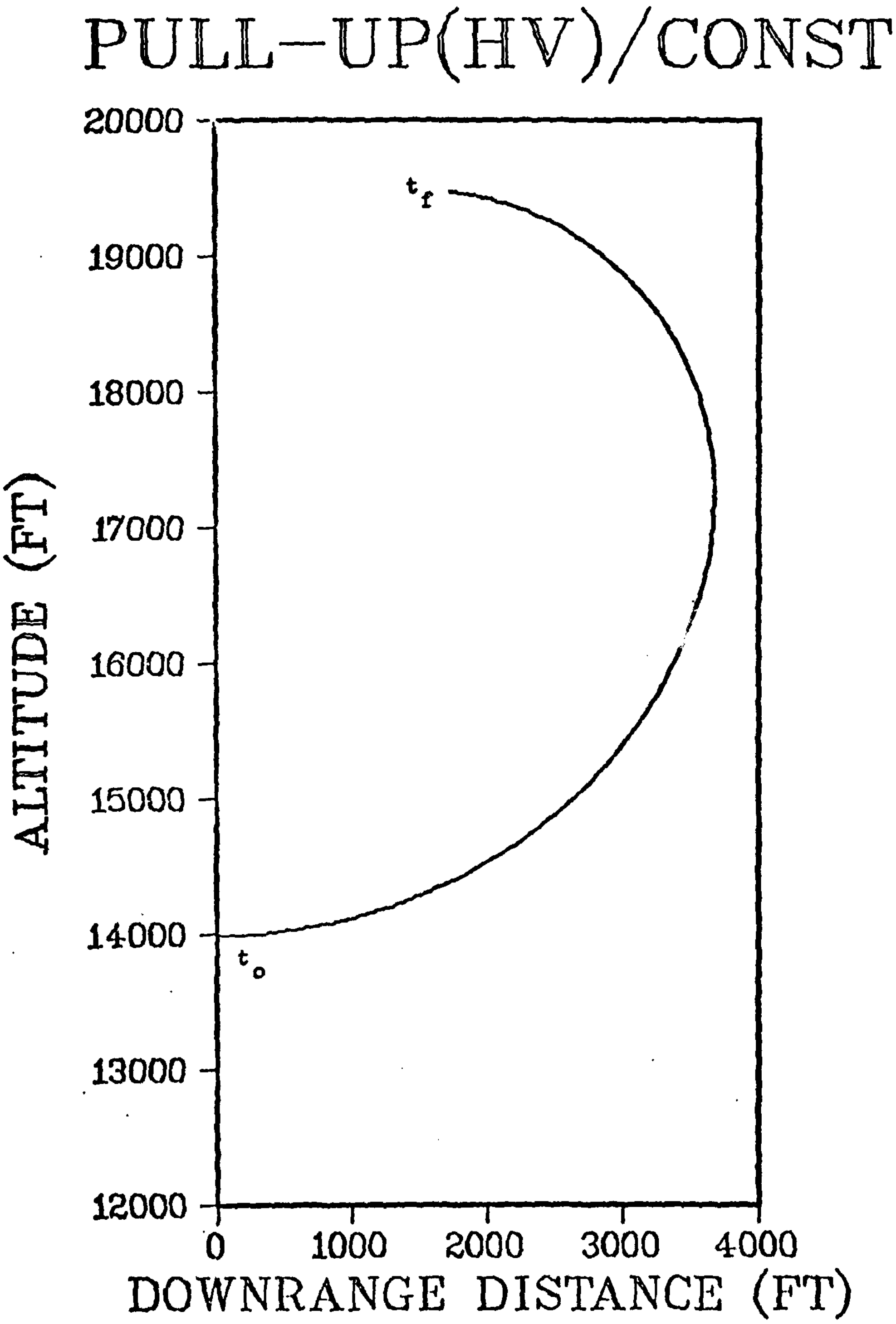

Figure 8b. Altitude vs. Downrange Distance

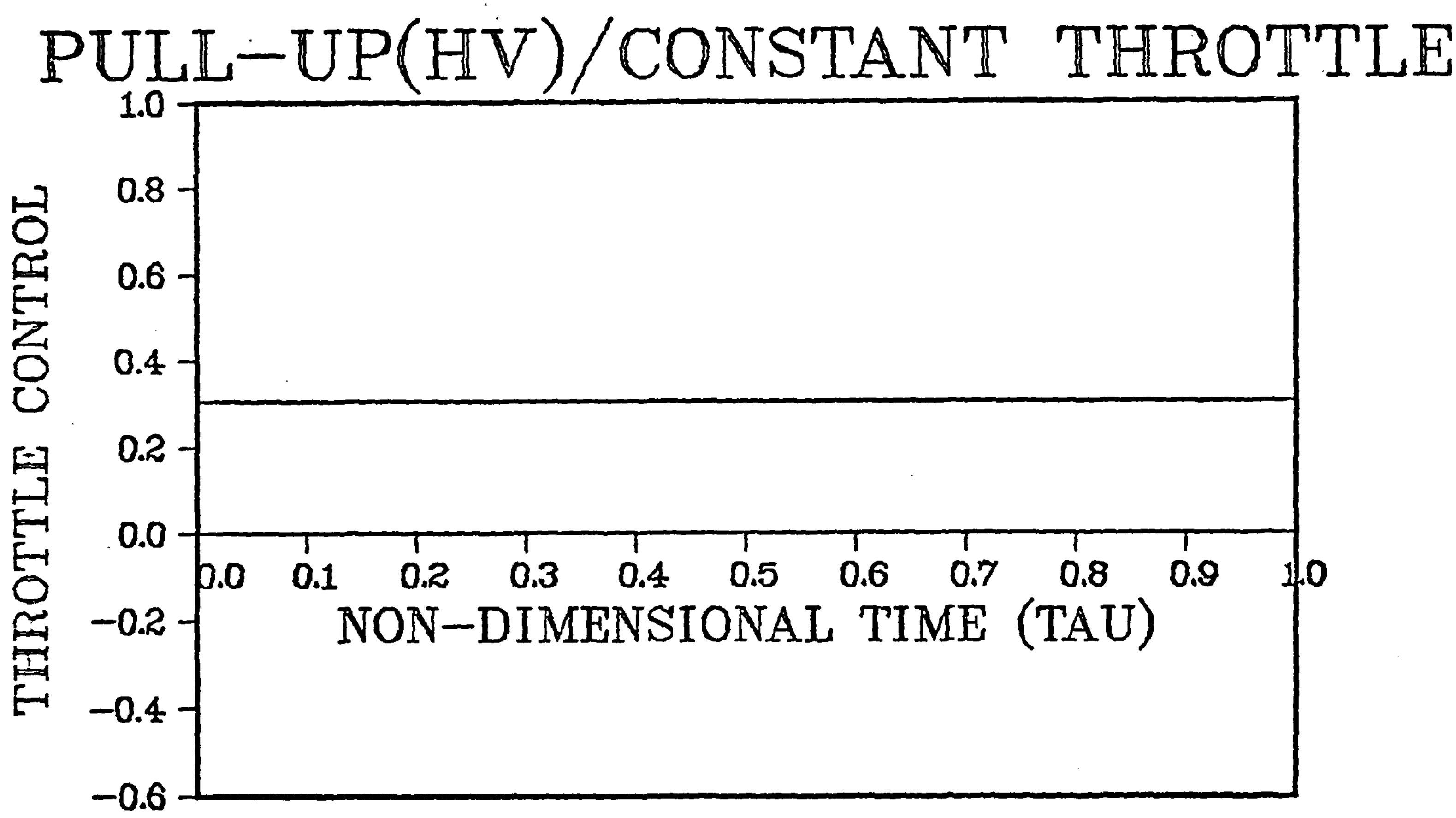

Figure 8c. Throttle Control vs. Non-Dimensional Time

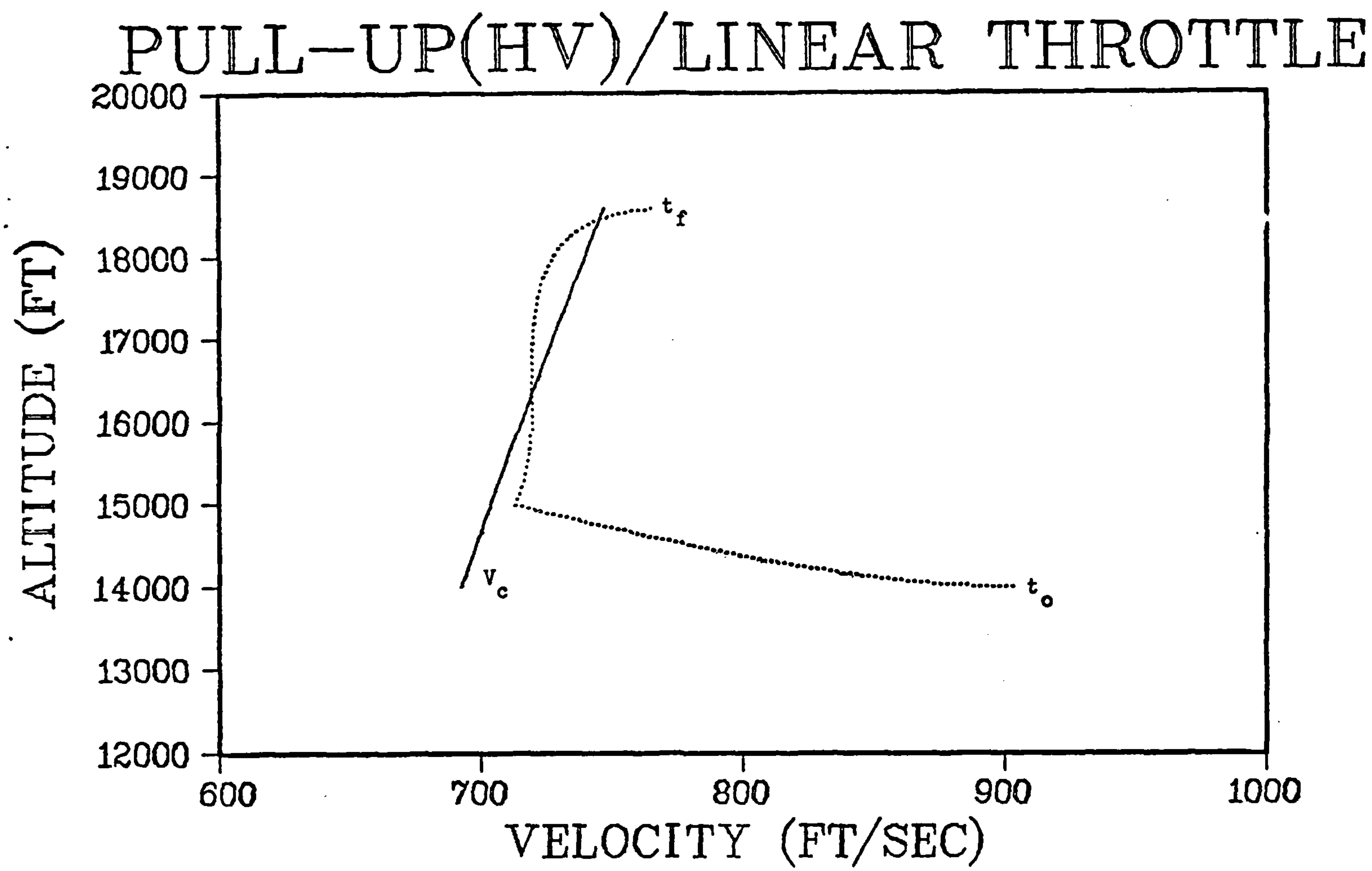

Figure 9a. Altitude vs. Velocity

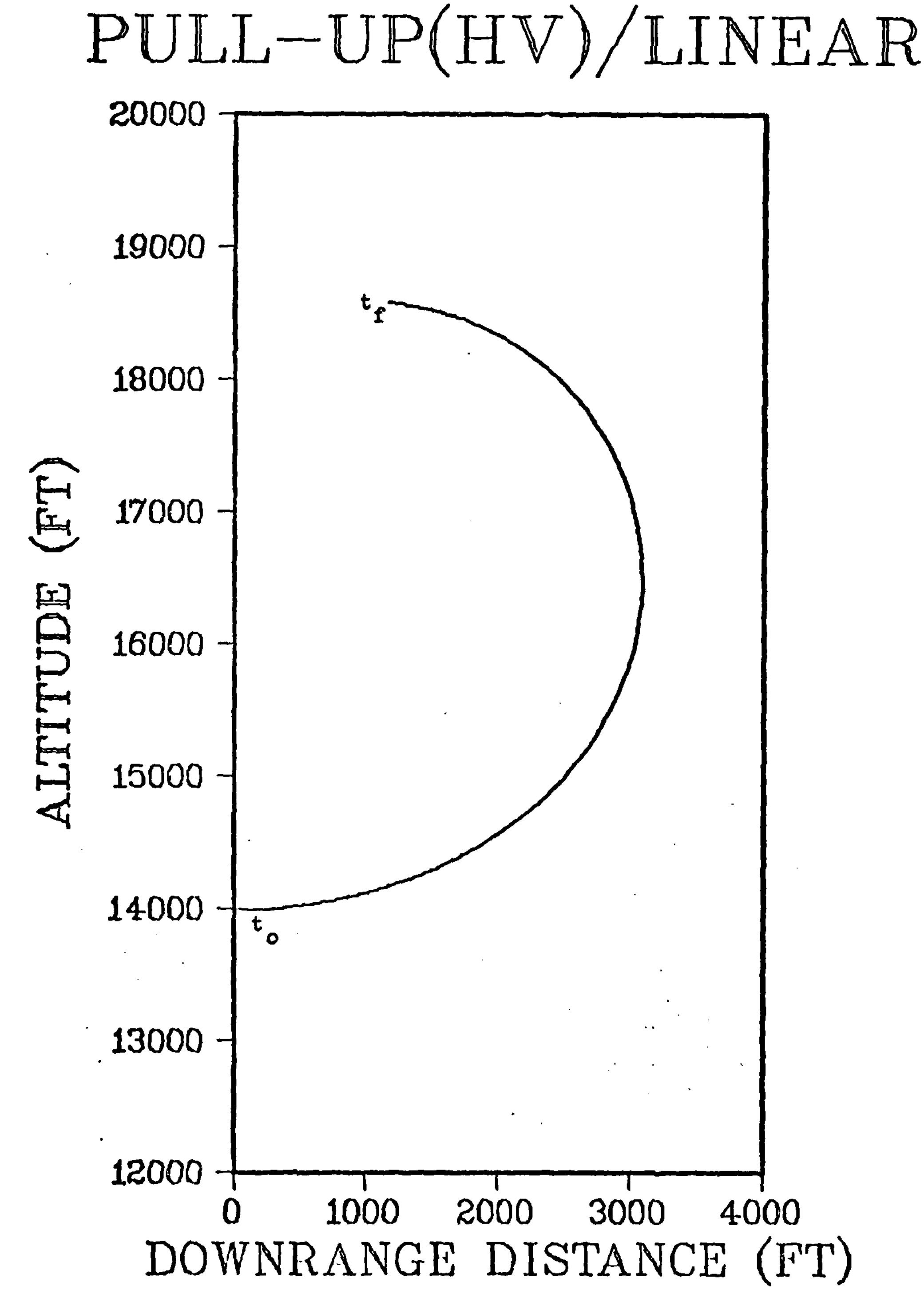

Figure 9b. Altitude vs. Downrange Distance

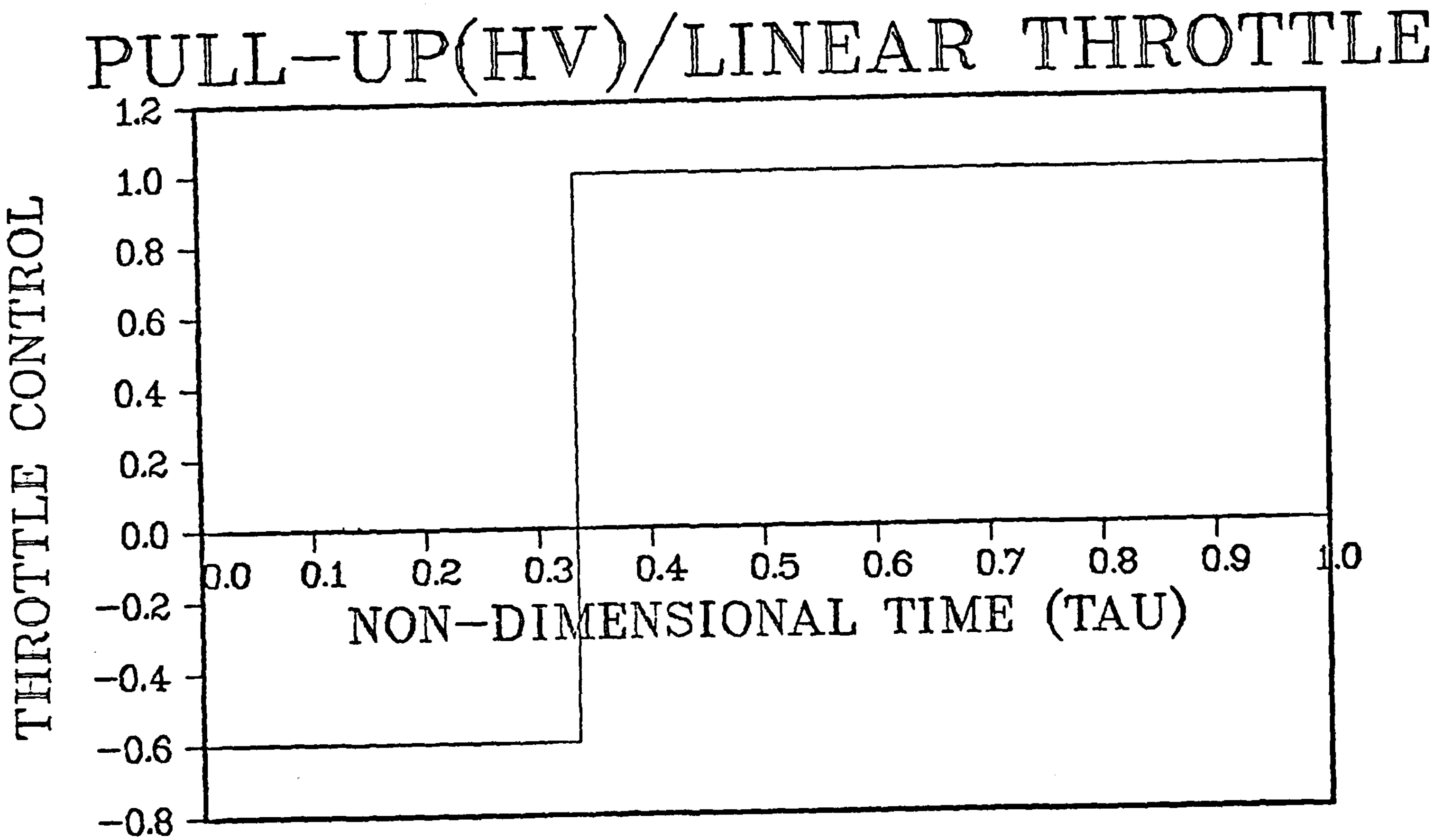

Figure 9c. Throttle Control vs. Non-Dimensional Time

MANEUVER	THROTTLE CONTROL	h_i (ft)	V_i (fps)	x_i (ft)	h_f (ft)	V_f (fps)	x_f (ft)	E_i (ft)	E_f (ft)	E_f-E_i (ft)	t_f (sec)
SPLIT-S (LV)	Constant	13990	621	0	10069	649	-335	19991	16623	-3368	9.631
	Linear	13990	621	0	10001	625	-390	19991	16080	-3911	9.554
	Quadratic	13990	621	0	10041	677	-301	19991	17173	-2818	9.278
	Cubic	13990	621	0	10053	644	-289	19991	16507	-3484	9.271
SPLIT-S (HV)	Constant	13990	903	0	8636	651	269	26679	15231	-11448	10.782
PULL-UP (LV)	Constant	13990	621	0	18257	747	249	19991	26940	6949	9.776
CASE 4		13990	621	-	17297	728	-	19991	25544	5553	9.554
PULL-UP (HV)	Constant	13990	903	0	19523	689	1541	26679	26910	231	11.152
	Linear	13990	903	0	18585	771	1007	26679	27835	1156	10.171
CASE 5		13990	903	-	17421	783	-	26679	26961	282	10.605

Table 6. Summary of Results

$$E = \left(h + \frac{V^2}{2g}\right) \text{ft} \tag{71}$$

where g is the gravitational acceleration at the initial altitude ($g=32.131$ ft/sec^2). The best overall minimum time to turn is the low velocity split-s maneuver with cubic throttle control. The final time is 9.271 sec; however there is a specific energy loss of 3484 ft. The minimum specific energy loss for this maneuver is with the quadratic control. Here the loss is limited to 2818 ft, a 19% improvement over the cubic throttle. Due to the minimal difference in turning times between the quadratic and the cubic throttle controls, it is evident that the most efficient way to perform the low velocity split-s is with quadratic throttle control. The only other maneuver where a choice can be made for the relative efficiency of the turn is the high velocity pull-up maneuver, and it is clearly evident that the linear throttle case is superior. In each of the two remaining cases, high velocity split-s ($t_f=10.782$ sec) and the low velocity pull-up ($t_f=9.776$ sec), the specific energy results were an 11,448 ft loss and a 6949 ft gain,respectively.

VI. Conclusions and Recommendations

Based on the results as listed in Table 6, a number of conclusions may be made regarding turning maneuvers performed in the vertical plane.

1. Trajectories restricted to the vertical plane gave different results, and in at least one case better results, than those not so constrained.
2. For the split-s maneuver, thrust reversal is beneficial in reducing the minimum time to turn regardless of whether the aircraft's initial velocity is above or below corner velocity.
3. For the pull-up maneuver, thrust reversal is beneficial in reducing the minimum time to turn only if the aircraft initial velocity is above corner velocity.
4. Although the split-s maneuver minimizes turning time, comparison of pull-up and split-s maneuvers in the vertical plane shows that for the split-s maneuver, specific energy is severely penalized when minimum turning time is desired.

Comparisons are made to the results of previous work, Case 4 and Case 5, in Ref (2). Comparison of the low velocity pull-up maneuver to Case 4 shows that although the pull-up required 2% more time to complete the maneuver, it provided for a 25% improvement in specific energy. Similar comparison of the high velocity pull-up with linear throttle control to Case 5 showed that the pull-up maneuver improved turning time by 4% and improved specific energy by 300%. As evidenced by these comparisons and the data summarized in Table 6, it can be stated that trajectories constrained to the vertical plane can be optimal minimum time to turn solutions. It can also be stated that thrust reversal can be used for minimum time turns with resulting increases in specific energy.

There are two recommendations that can be made regarding further study of the minimum time to turn problem. These recommendations are:

1. The closeness of the turning times indicate that for this aircraft it doesn't make any difference in which plane (straight lines as mapped in the y-h plane) the minimum time turn is performed, each of the cases have turned out fairly close. It would be interesting to determine the minimum turning times for all of the different planes.
2. The development of a set of aircraft equations of motion which can transition from vertical plane motion to 3-D motion with no problems is required in order to do accurate performance studies of high performance aircraft. The use of quaternions which are used in inertial navigation might be a big step in this direction.

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The objective of this study is to find the throttle controls and trajectories which yield the minimum turning times for a high performance aircraft with thrust reversal capability. The aircraft remains in the vertical plane allowing only pull-up and split-s maneuvers. A second-order parameter optimization method coupled with the suboptimal control approach is used to solve the minimum time to turn problem.

The results of the study found that trajectories restricted to the vertical plane gave different results, and in at least one case better

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) results, than those not so constrained. The results also indicate that depending on the maneuver performed, thrust reversal is beneficial in reducing the minimum time to turn regardless of whether the aircraft's initial velocity is above or below corner velocity. Finally, the results demonstrate that thrust reversal can be utilized for minimum time turns with resulting increases in specific energy.